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LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT FOR THE SESSION 1890–1 (NOV. 14TH, 1890, TO NOV. 11TH, 1891).

CASH ACCOUNT.

Dr.

				£.	s.	d.
Balance from 1889–90:—						
General Fund	£53	17	3			
De Morgan Medal Fund ...	1	8	10			
				5	6	1
Interest on Capital—						
Lord Rayleigh's Fund ...	53	3	1			
Lieut.-Col. Campbell's Fund	15	0	4			
Life Compositions Fund ...	27	9	3			
Invested Surplus Fund ...	13	6	8			
De Morgan Medal Fund ...	2	17	8			
				111	17	0
12 Entrance Fees				12	12	0
91 Subscriptions—						
3 for 1887–8	£3	3	0			
9 for 1888–9	9	9	0			
36 for 1889–90	37	16	0			
38 for 1890–1	39	18	0			
5 for 1891–2	5	5	0			
				95	11	0
Sales of *Proceedings*				106	10	3
4 Life Compositions...				42	0	0
				£423	16	4

Cr.

				£.	s.	d.
Printing *Proceedings*, &c.				271	9	0
Purchase of Journals, &c.				7	15	0
Binding				6	16	0
Postage, Stationery, and Sundries				17	10	1
Rent				10	0	0
Coals and Gas				1	10	0
Attendance—						
British Association ...	£1	5	0			
Mr. Stewardson	10	10	0			
				11	15	0
Life Compositions Fund—						
Purchase of £43. 18*s*. 0*d*. 2¾ per Cent. Consols				42	0	0
Balance at Bank—						
General Fund	£50	14	9			
De Morgan Medal Fund	4	6	6			
				55	1	3
				£423	16	4

Audited and found correct,

December 9th, 1891. (Signed) GEORGE HEPPEL.

				£.	s.	d.
Cash at Bank				50	14	9
132 Subscriptions owing—						
1 for 1885–6	£1	1	0			
1 for 1886–7	1	1	0			
5 for 1887–8	5	5	0			
12 for 1888–9	12	12	0			
36 for 1889–90	37	16	0			
77* for 1890–1	80	17	0			
				138	12	0
				£189	6	9

				£.	s.	d.
5 Subscriptions for 1891–2 paid in advance				5	5	0
6 Subscriptions struck off as irrecoverable—						
1 for 1885–6	£1	1	0			
1 for 1886–7	1	1	0			
1 for 1887–8	1	1	0			
1 for 1888–9	1	1	0			
1 for 1889–90	1	1	0			
1 for 1890–1	1	1	0			
				6	6	0
Balance				177	15	9
				£189	6	9

* Viz., 117, the number of Subscribers for 1890–1, less 2 Subscriptions paid in advance in 1889–90, and 38 paid in 1890–1.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	4	6	6	Balance	4	6	6

CAPITAL ACCOUNT.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
General Fund—							
1. Lord Rayleigh's Fund	1000	0	0	870	0	0	Guaranteed Five per Cent. Great Indian Peninsula Railway Stock.
2. Lieut.-Colonel Campbell's Fund	499	7	5	385	0	0	Guaranteed Four per Cent. London & North Western Railway Consolidated Stock.
3. Life Compositions Fund	1036	0	0	1057	1	3	2¾ per Cent. Consols, Account A.
4. Invested Surplus Fund	500	0	0	496	19	6	,, ,, ,, ,, B.
De Morgan Medal Fund	103	5	3	104	19	8	,, ,, ,, ,, C.

Audited and found correct,

December 9th, 1891. (Signed) GEORGE HEPPEL.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXII.

FROM NOVEMBER, 1890, TO NOVEMBER, 1891.

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1891.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXII.

TWENTY-SEVENTH SESSION, 1890—1891.

November 13*th*, 1890.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. E. J. Brooksmith was admitted into the Society.

The President informed the members of the loss the Society had sustained by the recent death of Dr. A. J. Ellis, F.R.S., and gave a short sketch of his mathematical and other work.

The President, rising again, indicated, in detail, the grounds upon which the Council had unanimously awarded the De Morgan Medal to Lord Rayleigh, in June, 1890.* He then presented the Medal to Lord Rayleigh, who expressed his thanks to the Society for their award.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Mr. S. Roberts, seconded by Prof. M. J. M. Hill, and carried unanimously.

At the request of the Chairman, Mr. G. Heppel consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of the members during the Session had been 200, now reduced to

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* This Address is published *in extenso* in *Nature* (Nov. 27th, 1890), Vol. XLIII., pp. 83–85.

197, by the death of Dr. Ellis, and the striking off the names of defaulters. The number of compounders was 86.

The Society had to regret the loss by death of Dr. A. J. Ellis, F.R.S., who was elected a member, June 19th, 1865, and served on the Council during the Sessions 1866–7, 1867–8.

The following communications had been made or received :—

Isoscelian Hexagrams: Mr. R. Tucker.

On Euler's ϕ-function: Mr. H. F. Baker.

On the Extension and Flexure of a Thin Elastic Plane Plate: Mr. A. B. Basset.

On the Radial Vibrations of a Cylindrical Elastic Shell: Mr. A. B. Basset.

Note on a Quaternary Group of 51840 Linear Substitutions: Dr. Morrice.

Note on a Plane Cubic and a Conic: Mr. R. A. Roberts.

On the Flexure of an Elastic Plate: Prof. H. Lamb.

On the Correlation of Two Spaces, each of Three Dimensions: Dr. Hirst.

On the Deformation of an Elastic Shell: Prof. H. Lamb.

On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points: Mr. A. B. Kempe.

On the Square of Euler's Series: Dr. Glaisher.

Complex Multiplication Moduli of Elliptic Functions for the Determinants -53 and -61: Prof. Mathews.

Concerning Semi-invariants: Mr. S. Roberts.

On Class-Invariants: Prof. Mathews.

On the Satellite of a Line relatively to a Cubic: Mr. Walker.

On the Properties of some Circles connected with a Triangle formed by Circular Arcs: Mr. Lachlan.

On a Theorem relating to Bicircular Quartics and Twisted Quartics: Mr. Lachlan.

On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$: Prof. Mathews.

On the Genesis of Binodal Quartic Curves from Conics: Mr. H. M. Jeffery.

On Simplicissima in Space of n Dimensions: Mr. W. J. C. Sharp.

On the Function which denotes the Excess of the Divisors of a Number which $\equiv 1$, mod. 3, over those of a Number which $\equiv 2$, mod. 3: Dr. Glaisher.

A Table of Complex Multiplication Moduli: Prof. Greenhill.

Note on the Simultaneous Reduction of the Ternary Quadric and Cubic to the Forms $Ax^2+By^2+Cz^2+Dw^2$, $ax^3+by^3+cz^3+dw^3$: Mr. Walker.

Ether-Squirts: Prof. K. Pearson.

Note on the Imaginary Roots of an Equation: Prof. Cayley.

Perfect Numbers: Major Macmahon.

The Relation of Distortion in Prismatic Images to Dispersion: Dr. Larmor.

Rotatory Polarization: Dr. Larmor.

Some Properties of Conormal Points on a Parabola: Mr. Tucker.

On the Expression of the Square Root of a Quartic as a Continued Fraction: Prof. Mathews.

On Modular Equations: Mr. R. Russell.

On certain Concomitants of a System of Conics and Quadrics, and on the Calculation of the Covariant S of the Ternary Quartic: Mr. A. R. Johnson.

The same Journals had been subscribed for and the same exchanges made as in the preceding Session. Further volumes of the "Collected Papers" of Prof. Cayley had been received, and, in addition, "The Scientific Papers" of Prof. J. Clerk Maxwell had been presented by the Syndics of the Cambridge University Press.

The meeting next proceeded to the election of the new Council.

The Scrutators (Mr. R. A. Sampson and Dr. G. G. Morrice) having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, Prof. Greenhill, F.R.S.; Vice-Presidents, Dr. J. Larmor, Major Macmahon, R.A., F.R.S., Mr. J. J. Walker, F.R.S.; Treasurer, Mr. A. B. Kempe, F.R.S.; Secretaries, Messrs. M. Jenkins and R. Tucker. Other Members: Messrs. A. B. Basset, F.R.S., E. B. Elliott, Dr. J. W. L. Glaisher, F.R.S., Mr. J. Hammond, Dr. T. A. Hirst, F.R.S., Messrs. R. Lachlan, C. Leudesdorf, A. E. H. Love, and S. Roberts, F.R.S.

Prof. Greenhill, having thanked the Society for the honour conferred upon him, called upon Mr. Walker to read his Address "Of the Influence of Applied on the Progress of Pure Mathematics." On the motion of Mr. Kempe, seconded by Lord Rayleigh, it was carried by acclamation that the Address, with the author's permission, be printed in the *Proceedings*. Mr. Walker then thanked the members present, and expressed his pleasure in acceding to their request.

The following communications were made:—

Spherical Harmonics of Fractional Order: Mr. Sampson.

Proofs of Steiner's Theorem relating to Circumscribed and Inscribed Conics: Prof. Mathews.

On an Algebraic Integral of Two Differential Equations: Mr. R. A. Roberts.

Some Geometrical Constructions: Mr. Osher Ber. (Communicated by Prof. M. J. M. Hill.)

On the Analytical representation of Heptagrams: Prof. L. J. Rogers.

The following presents were received:—

Cabinet Likeness of Dr. Story and Carte de Visite Likeness of Mr. Walker for the Society's Album.

"Educational Times," for November.

"The Collected Mathematical Papers of Professor Cayley," Vol. III. (two copies), 4to; Cambridge, 1890.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. LIII., 8vo; London, 1890.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Fourth Series, Vol. III., 8vo; Manchester, 1890.

"Annals of Mathematics," Vol. V., No. 3, 4to; Washington, 1890.

"Washington Observations," 1884, 4to; Washington, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stücke 9 und 10.

"Bulletin des Sciences Mathématiques," Tome XIV., Oct., 1890.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," No. 115; Firenze, 1890.

"Reale Instituto Lombardo di Scienze e Lettere—Rendiconti," Serie II., Vols. XXI. and XXII.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," 22 Heft; München, 1890.

"Annali di Matematica," Serie II., Tomo XVIII., Fasc. 3.

"Prace Matematyczno-Fizycne," Tom. II., Zeszyt II.; Warsaw, 1890.

"Journal für die reine und angewandte Mathematik," Band 107, Heft II.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 5; Roma, 1890.

"Atti della Reale Instituto Veneto di Scienze, Lettere, ed Arti," Tomo 38 (T. I., Ser. VII., Disp. 1–9); Venezia, 1889–90.

"The Journal of the College of Science, Imperial University, Japan," Vol. III., Part IV.; Tokyo, 1890.

"American Journal of Mathematics," Vol. XIII., 1.

Of the Influence of Applied on the Progress of Pure Mathematics. (*Presidential Address.*)

[*Read November* 13*th*, 1890.]

The subject I have chosen, as outgoing President, for a brief address to this Society, seems to me a not inappropriate sequel to the presentation which has just occupied us. If treated exhaustively it would require both extensive research and much more space than is now at my disposal; but, by dwelling at any length on a few epoch-forming steps in advancement only, I hope to keep myself within reasonable bounds, and to find time for some digressions suggested by and arising out of the matter proper of my subject. Much of what I shall have to say might have found place in a general history of the progress of mathematics; but it has not, as far as I am aware, entered into the plan of any such work to interweave with its narrative the kind of comments I have in view to make. These, I hope, may be of some interest in themselves, and will serve to relieve what would, otherwise, be a bald enumeration of instances of impulse given to one branch of mathematics by the other.

Whether or not the conjecture of Herodotus (II., 19) was well founded, that the origin of Geometry was to be looked for in the need—for adjustment of the land-tax—of remeasuring plots which had been diminished by the inundations of the Nile, it is generally admitted that the study and perfecting of Plane and Spherical Trigonometry sprang from the necessity of their application to Astronomy. To the same cause is to be assigned the calculation and publication of those numerous tables, first of chords, then of sines and tangents in parts (primitively, sexagesimal) of the radius, from that of Ptolemy in the second, to the "Canons" which appeared in the sixteenth century.

But, more especially, the need felt of curtailing the labour of solving plane and spherical triangles in astronomical calculations, by means of natural sines and tangents, led to the invention and computation of logarithmic sines, a table of which[*] formed the "Canon Mirificus" (1614) of Neper, or Napier[†], of Merchistoun (1555–1617), the use whereof was illustrated by application to his "Rules" for the solution of right-angled spherical triangles; as well as to other cases. These logarithms, being "tabular," are not, of course, indices of powers of a "base"; and, therefore, the expression so commonly met, "base of Napier's logarithms," is conventional only and inexact. In fact, for a century or more after Napier's discovery, logarithms were defined without specific reference to a "base"; his own definition being as follows:—

Suppose two lines AO and $a\omega$—the latter finite and representing the radius 10^7, the former of indefinite length—to be described by points B, β, starting simultaneously from A, a respectively with equal velocities (u), the velocity of B continuing uniform, but that of β following the law

---

[*] Prefixed to the Tables are the lines:—

"In Logarithmos."
"Quæ tibi cunque sinus, tangentes atque secantes,
Prolixo præstant atque labore gravi,
Absque labore gravi et subito tibi candide lector,
Hæc Logarithmorum parva tabella dabit."

[†] So printed in the "Anatomy of Melancholy" (1621): P. II., S. ii., M. 4.

$$v = u \,.\, \beta\omega / \alpha\omega\,;$$

then C, γ being the positions of B, β at any the same instant, "the number representing the length of AC will be the logarithm of (that representing) the sine $\gamma\omega$..., whence the logarithm of the sine of the whole 10^7 will be zero."

Now, if $$\alpha\omega = r, \quad \gamma\omega = x,$$

for the movement of β at the end of the time t,

$$dx = -\,xu\,dt/r,$$

which, being integrated between the limits a and x, gives

$$ut \text{ or } AC = a \log_e (a/x)\,;$$

i.e., n being the number representing x,

$$\text{Nap. } \log n = 10^7 \log_e (10^7/n).$$[*]

Hence, to convert Napier's logs. into logs. to base e, subtract the former, divided by 10^7, from $\log_e 10^7$.

A specimen line of Napier's "Canon Mirificus," with heading and tail of page, is

Gr 11 min.	Sinus		Logarithmi	Differentiæ	Logarithmi		Sinus	
*	*		*	*	*		*	*
15	1950903		16342924	16148908	194016		9807853	45
*	*		*	*	*		*	*
						78		

From this $\quad$ Nap. log. sine $11^\circ 15'/10^7 = 1{\cdot}6343$ q.p.,

and $\quad \log_e 10^7 - 1{\cdot}6343 = 14{\cdot}4846 = \log_e 1951000,$

---

[*] This relation is pointed out by Dr. Glaisher, Art. "Logarithms," *Enc. Brit.*, 9th ed.; but the deduction of it from Napier's own definition may be of interest.

which last number is (q.p.) sine 11° 15′ to radius 10^7. "Antilogarithm" was defined by Napier as "logarithm of the complement."

Briggs began his "Arith. Log." (1624) with the definition: "Logarithmi sunt numeri qui, proportionalibus adjuncti, æquales servant differentias." Then, 10 and 1 being fixed on as the two "principal numbers,"—viz., those whose logarithms are to be 1 and 0 respectively,—he proceeds to show that the logarithms of all numbers are either equal or proportional to the indices of powers of 10. Thus a "base" is substantially, though not explicitly, recognised.

Cotes (*Phil. Trans.*, Vol. XXIX.) explained the theory of the "modulus" of a system of "measures of ratios,"—so he names those quantities of which "logarithms are the numerical forms,"—and to the number 2·718... he gave the name "Modular Ratio," viz., "that determinate and constant number, the 'measure' of whose ratio to unity is the 'modulus' of the system employed."

But the modern treatment of logarithms as explicitly the "indices of powers" appears to have originated* with Euler (1707–83), who writes in the Preface (p. ix.) to the *Anal. Inf.* (Lausanne, 1748): "Postquam autem a potestatibus ad quantitates exponentiales essem progressus quæ nihil aliud sunt nisi potestates, quarum exponentes sunt variabiles; ex earum conversione maxime naturalem et fecundam Logarithmorum ideam sum adeptus."

And in the same work, Vol. I., p. 73, ..."Quod si fuerit $a^z = y$, erit $z = ly$ ... Supponit ergo doctrina logarithmorum numerum certum constantem loco a substituendum, qui propterea *basis* Logarithmorum vocatur."

To Euler also is due the introduction of "e" into Analysis, as an important and fundamental constant. Having defined (*ib.*, p. 85) k by the equation

$$a^{\omega} = 1 + k\omega,$$

ω being infinitely small, and proved (*ib.*, p. 88)

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\,.\,2} + \ldots,$$

$$k = \frac{a-1}{1} - \frac{(a-1)^2}{1\,.\,2} + \frac{(a-1)^3}{1.2.3} - \ldots,$$

---

* In a paper communicated (after the author's death) to the R. S., which will be found in the *Phil. Trans.*, 1771 (Vol. LXI., p. 455), W. Jones introduced the definition of logarithms, viz., "indices of powers of the same radical number," as novel.

he continues: "Ponamus autem brevitatis gratia pro numero hoc 2·718 ... constanter literam *e*, quæ ergo denotabit *basim* Logarithmorum naturalium seu hyperbolicorum, cui respondet valor literæ $k = 1$; sive hæc litera *e* exprimet quoque summam hujus seriei

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\&\text{c. in infinitum."}$$

Euler's choice of this letter was probably determined simply by its being the consecutive vowel to *a*,[*] which he had employed as the general base of any system of logarithms. We, however, may remember that it is the initial letter of the great mathematician's name, and piously adhere to its use "in memoriam,"[†] avoiding the variation *e* which has been used by some English writers. The great importance of this constant in the Integral Calculus is then noticed (*ib.*, p. 93), and the expressions for sine v and cosine v in terms of exp. $v\sqrt{-1}$ are immediately (*ib.*, p. 104) deduced—a fact which I notice because I think that the introduction of these exponentials into Analysis is incontestably due to Euler, though attributed by many English writers[‡] to De Moivre. In the ***Misc. An.*** (1734), the latter gives as his first "Lemma" the formula

$$x = \frac{z}{2}+\frac{1}{2z},$$

when $$z = \sqrt[n]{l+\sqrt{ll-1}},$$

l and x being the cosines of two angles in the ratio of n to 1; and the only application made of this formula is to the resolution of the trinomial equation

$$1-2lz^n+z^{2n} = 0$$

into the quadratics

$$1-2az+zz = 0,\quad 1-2bz+zz = 0,\ \ldots\ldots$$

as many in number as is that of the units contained in n. The exponential expressions for $\sin v$, $\cos v$ may, as is often shown, be derived from the Lemma, but the step is by no means so obvious as it was to the analyst who had been led to the series whose sum is represented by exp. z, and had before him (*ib.*, p. 99) the series for $\sin v$, $\cos v$.

---

[*] James Bernoulli, I believe, first introduced the "exponential" into Analysis, using this form "a^z."

[†] I quote these words from a paper in the *Royal Soc. Proc.* (Vol. xxxi., p. 381), by our lately deceased member, A. J. Ellis, in a parallel case.

[‡] *E.g.*, De Morgan, Art. "De Moivre" in *P. Cyclop.* and *Eng. Dict. of Biography;* also *Diff. and Int. Calculus*, pp. 129, 130.

I return from this digression to my more immediate subject. What is now known as the "Logarithmic Spiral," is a curve to the discovery of which Des Cartes (1596–1650) was led by an investigation of the path of a body falling, under the action of a constant force directed to a fixed centre, but not towards that centre.[*] The characteristic properties of this curve Des Cartes found to be that its length between the pole and any point is proportional to the length of the straight line connecting them, and that of cutting this connector at a constant angle.[†] Either property follows immediately from the other, but no proof is offered by Des Cartes. When subsequently studied by other geometers, and more particularly by James Bernoulli[‡] (1654–1705)—as the limit of the Loxodromic curve on the sphere, *i.e.*, the curve which cuts all meridians at a given angle, the solution of a problem in navigation—the property from which it gets its modern name[§] was observed, viz., that rays which intercept on a circumference, concentric with the force, arcs which, measured from a fixed ray, are in arithmetic progression, intercept on the curve corresponding arcs which are in geometric progression. The former series of arcs are therefore the "measures" (Cotes) of the latter, or, the two series representing numbers, those of the former will be the "logarithms" (Briggs) of those of the latter—in some system, the modulus of which will be a function of the angle of the spiral. The recent employment of this curve in "Graphical Involution"[‖] is the repayment of the debt which Pure owes to Applied Mathematics in respect of it.

The "Ovals" of Des Cartes, about which so considerable a mass of literature has gathered, were brought under that geometer's notice by his researches in Dioptrics, as pointed out by Chasles.

Another curve now usually defined by means of an exponential equation,—James Gregory's (1638–1675) "Logarithmic Curve,"—was represented by him[¶] through a relation between the ordinate y

---

[*] Lettres de Des Cartes (73) au Rev. P. Mersenne, t. i., p. 414.

[†] *ib.* (74), p. 436.

[‡] *Acta Erud.*, 1691. The epitaph which he "would gladly have inscribed on his tomb," is inexactly given by many English writers: it was "Eadem *numero* mutata resurget."

[§] John Bernoulli generally refers to it as "Fratris Spiralis."

[‖] Culmann, *Graphische Statik*.

[¶] *Geom. Pars Univ.* (1668). The name "Logarithmic" was given by Huyghens, *Sur la Cause de la Pésanteur* (1690).

and abscissa x which would be equivalent to

$$(y : y')^{x''-x} = (y'' : y)^{x-x'},$$

(x, y) being a variable point, (x', y'), (x'', y'') fixed points on the curve. The above equation is substantially the same, of course, as the more modern form, but contains no explicit reference to the "base" of a system of logarithms. Accordingly, in proving that "one ordinate being taken as unity and others representing numbers, the intercept on the asymptote between unity and any number would be the logarithm of that number," he simply shows that if the ordinates are equidistant they will form a geometric progression.

If Probabilities may be considered a branch of Applied Mathematics, we owe to this theory the introduction into pure analysis of the "numbers of Jas. Bernoulli,"* who first appears to have used "exponentials," or quantities with a variable exponent. By him the Numbers were defined as the coefficients of the successive terms, after the first two, of the expansion of

$$2\int n^c\,dc,$$

in descending powers of n, when divided by the first, third ... binomial coefficients

$$c,\ \frac{c\,.\,c-1\,.\,c-2}{1\,.\,2\,.\,3}\ \ldots,$$

and multiplied by the natural numbers 1, 2, ... respectively; viz., if

$$\int n^c\dagger \propto \frac{1}{c+1}\,n^{c+1} + \frac{1}{2}\,n^c + \frac{c}{2}\,An^{c-1}$$

$$+\frac{c\,.\,c-1\,.\,c-2}{2\,.\,3\,.\,4}\,Bn^{c-3} + \frac{c\,.\,c-1\,\ldots\,c-4}{2\,.\,3\,\ldots\,6}\,Cn^{c-5} + \ldots,$$

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[*] Euler refers to this expression as become customary.

[†] Thus printed; but perhaps in the MS. it might have been written In^c in the sense of "the expression of which $n^c dc$ is the differential." In the P.S. of a letter to John Bernoulli, dated "1696 Mart.," Leibnitz writes: "Velim audire sententiam tuam, an probes adhiberi notam $\int$ pro summis, ... d pro differentiis ... ut idem sit $a : b$ quod $\frac{a}{b}$; id enim præsertim in typis commodum est, ne linearum spatium amittatur."

J. B., in a reply dated "Groningæ, 7 Aprilis 1696," states that he will use S hereafter in place of "quod nos antehac per Integrale et I denotare voluimus, quæ expressio jam passim invaluit."

The sign of equality used is that employed by Des Cartes.

then $A, B, C, D \ldots$ are the numbers in question. In the words of the author, "... A, B, C, D, &c., ordine denotant coefficientes, ultimorum terminorum pro

$$\int nn,\quad \int n^4,\quad \int n^6,\quad \int n^8,\quad \text{\&c.},$$

nempe $$A \propto \frac{1}{6},\quad B \propto -\frac{1}{30},\quad C \propto -\frac{1}{42},\quad D \propto -\frac{1}{30}.$$

Sunt autem hi coefficientes ita comparati ut singuli cum cæteris sui ordinis coefficientibus complere debeant unitatem. Sic D valere diximus $-\frac{1}{30}$, quia (posito $c \propto 8$),

$$\frac{1}{9}+\frac{1}{2}+\frac{2}{3}-\frac{7}{15}+\frac{2}{9}-\frac{1}{30} \propto 1.$$

It is superfluous to remark that different multiples of these numbers have been called "Bernoulli's Numbers" by different writers. It might be wished that the original symbols $A, B, C, D \ldots$ should be called by the discoverer's name, and B's, marked, used for modifications of them; but what is to be observed is that in the original definitions there is no use of the base e.

Thus fruitful of results in Pure Mathematics was the invention communicated to the scientific world through the "Canon Mirificus," aiming only though it did at relieving the labour of astronomical calculations; and of which the author could write so modestly—nay, apologetically—in the "Admonitio" which follows his brief description of the Tables, as his reason for deferring the explanation of their construction: "ut prælibatis prius usu et rei utilitate cætera aut magis placeant post hoc edenda aut minus saltem displiceant silentio sepulta."

In tracing the impulse given to Pure Mathematics by Napier's great invention, I have been brought to the middle of the eighteenth century. Returning to the seventeenth, the next important case of my subject which presents itself is the stimulus given to the study and progress of the Leibnitzian Calculus,* by its applicability—or rather indispensableness—to the solution of the physical problems, by the proposal of which the leading mathematicians of the period challenged one another to trials of skill and efficiency of methods.

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* I use this term (without any intention of pronouncing on the vexed question of priority in the first conception of the method) in the sense in which it is continually employed by the continental mathematicians, who derived all their acquaintance with its fundamental principles from Leibnitz's short, but classical, essay, which appeared in the year 1684 in the *Acta Eruditorum* (founded by himself and Otto Mencke in 1683). This little tract of some seven (467–73) pages of the small 4to of the *Acta*

James Bernoulli (1654–1706), recently returned from a visit to Geneva and France (1675–80), and now settled in Basel, and occupied with the Cartesian Philosophy, took up the study of the new Calculus with great ardour, and is justly credited with the establishment of the foundations of the Integral Calculus (his essays on which appeared in 1691), to which he was mainly led by considerations arising out of the solution of physical problems. In the *Nouvelles de la République des Lettres*, September, 1687, Leibnitz proposed* as a challenge the problem: "Trouver une ligne de descente dans laquelle le corps pesant descende uniformément et approche également de l'horizon en temps égaux"—adding: "L'Analyse de ces messieurs les Cartésiens le donnera peut-être aisément"—for vertical spaces, the geometrical solution of which by Huyghens appeared in the same Journal for the following month, and Leibnitz's synthetical solution by the method of his Calculus in the *Acta Erud.* for 1689; but in the *Acta* of the following year, James Bernoulli† (who had been appointed Professor of Mathematics at Basel in 1687) gave the first analytical solution of such a problem involving the solution of the differential equation

$$dy\sqrt{bby-a^3} = dx\sqrt{a^3},$$

and proposed as a return challenge: "Invenire quam curvam referat funis laxus et inter duo puncta libere suspensus." This problem was solved analytically by his younger brother, John‡ (1667–1748), who for the uniform catenary wrote:—"Igitur

$$dy = adx : \sqrt{xx+2ax}$$

æquatio est ad funicularem vulgarem, quæ quoniam nequit integrari indicio est curvam inter Mechanicas referendam esse." On which solution the editor of his works wrote: "Problema istius (*sc.* James

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contains merely proofs of the methods of differentiating the product and quotient of two functions of a single variable, and of obtaining the maximum and minimum values of a function of a single variable. It is difficult to view this germ of so great developments without emotion: "Nascitur exiguus, vires acquirit eundo." To trace the influence of the study of physical problems on this acquisition of strength is the object of the present section of my essay.

* At the conclusion of a letter on the Cartesian measure of force.

† The problem of the curve of uniform recession from, or approach to, a fixed point under the action of gravity, led to the introduction of the "Lemniscate" into Geometry, by James Bernoulli (*Op.*, p. 609).

‡ *Acta Erud.* 1691, with the results of solutions by James Bernoulli himself, Leibnitz and Huyghens.

Bernoulli) quod primum fere dubitantes Geometras usum Calculi Leibnitziani docuit." Huyghens, than whom no mathematician of the time was held in higher esteem by Leibnitz, as a staunch Cartesian, had not as yet abandoned geometrical methods. On his death in 1695, Leibnitz wrote to John Bernoulli: "Incomparabilem Hugenium obiisse haud dubie intellexisti. Quanta hæc sit jactura dici satis non potest, ob summum viri judicium cum maxima profundissimaque rerum notitia conjunctum." (P. S. to Ep. xv., dated "Hanovera, 29 Jul., 1695.")

Huyghens' books were sold by auction, and under date "Groningæ (to the Professorship of Mathematics at which place John Bernoulli had been appointed recently), 7 Aprilis, 1696," the latter wrote to Leibnitz that Huyghens' copies of the *Acta Erud.* for 1690, 3, 4, had been sent to him, with marginalia on what Leibnitz, Tschirnhaus, James Bernoulli and himself had published therein. In reply (Ep. xxvi.), Leibnitz, after thanking his correspondent, adds: "Initio parum favebat methodis nostris quod fructum earum sua amicorumve experientia nondum didicisset, sed postea rectius doceri cœpit, et cum in literis privatis ('votre merveilleux calcul') tum in schediasmate quodam inter Acta Lipsiensia publicato, candide pronuntiavit quod res est." And again, in Ep. xxviii.: "In Actis Eruditorum, 1693, p. 476, diserte commendat Calculi Differentialis (quo tunc feliciter usus erat) inventionem; *absque quo,* inquit, *vix est ut ad has subtilitates admitteremur.*" This is an eminent, and, no doubt, typical instance of the way in which the solution of dynamical problems led to the recognition of, and won converts to, the "wonderful calculus," which, whether from the prevalence of Cartesianism, or the inertia of habit, otherwise made very slow way. In one letter, John Bernoulli remarks that in the History of the Paris Academy of Sciences, 1700, no mention is made of the Calculus; nor is any other cultivator of it, but de l' Hôpital, known among the contemporary French mathematicians; while, as late as 1705, Octr. Leibnitz, writing to condole with John Bernoulli on the death of his elder brother (whom he immediately succeeded in the chair of Mathematics at Basel), declares that no German is proficient in the principles of the *Calculus*, Wolff* alone having some acquaintance therewith. Of our countrymen John Bernoulli had written in 1696 (Ep. XXIX.), on occasion of the appearance of a new edition of Wallis's (1613–1703) works: "Calculum Differentialem non ita laudat ut decet. Suos Anglos, pro more suo solito, mirum quantum extollat. Prae illis nil fere est quod alii fecerunt." It will be remembered that nothing had been printed of the rival method of fluxions in England before the appearance of the celebrated Scholium to Lemma 2, P. vii., Lib. ii., of the first edition of the *Principia* (1686).

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* This was J. Ch. von Wolff (1679–1754), more eminent as a moralist, theologian, and psychologist, than as a mathematician; Professor of Pure and Mixed Mathematics at Halle, then of Mathematics and Philosophy at Marburg. His "Elementa Matheseos Universæ" was long the principal text-book of pure and applied science in Germany; but he contributed little to its progress otherwise than as a teacher.

I have quoted largely from the *Commercium Epistolicum* of Leibnitz and John Bernoulli. It is full of interest, and perhaps opens out the mathematical side of the great German philosopher more completely than any other remains. Not but that physical, metaphysical, and moral topics occasionally enter; yet the subjects are mainly mathematical. The correspondence arose out of an application of the younger man to Leibnitz to recommend him to the Elector of Brunswick, Leibnitz in his reply saying how much pleasure it would give him to correspond on mathematical subjects with the applicant. As long as it lasted, it formed a sort of journal of the scientific lives of both the writers.

Lagrange's (1736–1813) Series has often been considered as an acquisition to Pure Mathematics due to Applied, so exactly does it seem suited to the solution of Kepler's Problem—the determination of the eccentric in terms of the mean anomaly;—an application, in fact, which followed almost immediately on its discovery. Those who have looked into the matter, however, know that it arose out of the logarithmic method of solving algebraic equations, having been given by Lagrange in a memoir—"Nouvelle méthode pour résoudre les équations littérales par le moyen des séries" (*Mém. de l'Ac. de Berlin*, t. XXIV.; *lu à l'Ac.* 18 Jan. et 5 Av. 1770),—as a formula for any rational function of a root of an equation obtained by induction. In another memoir, read Nov. 1 of the same year, "Sur le problème de Kepler," the application referred to is made, the relation

$$t = x + n \sin x$$

having been proved. Nor was Laplace's (1749–1827) extension of Lagrange's theorem due to any application in Physics, having, in fact, been discovered before Laplace had entered upon those great physical studies on which his reputation is mainly founded. The extension was published in the *Mém. de l'Ac. des Sciences* for 1777, when the author was only 28 years old, the title being "Sur l'usage du Calcul aux Différences Partielles dans la Théorie des Suites."

In ignorance, apparently, of the Memoir of Lagrange—though having made the fullest searches which occurred to him for any anticipation of his method—sixty years later, R. Murphy, of Caius, in the *Transactions of the Camb. Phil. Soc.* for 1831, reproduced as new the matter of Section II. of that Memoir—even to pointing out how Lagrange's Series might be derived by induction. This unconscious plagiarism escaped even the critical De Morgan, who could write of the logarithmic method of solving equations (*Diff. and Int. Cal.*, § 213, p. 328): "This theorem, given by Mr. Murphy, . . . is one of the most general and interesting contributions which Analysis has received for many years." The real novelty in Murphy's memoir was the expression given—in the form of a definite integral—for the "error" involved in stopping at any given term of Lagrange's Series. This is quite different in form from Popoff's expression (*C. R.*, 1861, pp. 795–8), which appears to be considered as the first attempt to sum up the "remainder." Yet, in some remarks on Lagrange's Series (*Diff. and Int. Calc.*, p. 171), our first President came "burningly" near the truth, though it does not seem to me that Lambert (J. H., 1728–77) in any way contributed to its discovery by his paper "Obss. Var. in Mathesin Puram," *Acta Helvetica*, III., pp. 128–168, which contains methods of approximating to the roots of equations, but nothing that could be considered as an anticipation of Lagrange's method.

The Series continued to be regarded and used as the best method of solving Kepler's Problem, until Bessel (1784–1846) introduced his treatment (*Abh. d. Berl. Ak.*, 1816–7) into an essay on the same problem. But his "Functions," in the form with which we are familiar, appear in a later memoir, "Planetarische Störungen aus der Bewegung der Sonne" (*Abh. d. Berl. Ak.*, 1824). These, the theory of which is an undoubted acquisition by Pure, due to Applied Mathematics, wrote Bessel (§ 8)—and it is perhaps to be regretted that the discoverer's notation has not been preserved "in memoriam"—"werde ich durch

$$\int \cos(h\epsilon - k\sin\epsilon)\,d\epsilon = 2\pi I_k^h$$

bezeichnen," the limits being 0 and 2π; while he uses J in the sense

$$\int_0^{2\pi} \frac{\cos(k\epsilon - h\sin\epsilon)}{1 - e\cos\epsilon}\,d\epsilon = 2\pi J_k.$$

It is of interest to remark that, for the development in a series of ascending powers of k, reference is made to the earlier memoir, and that he devoted § 11 to the consideration of his Functions as belonging to pure analysis, "examining them more closely" and deducing the well-known equation of the 2nd order "which defines I_k^i," together with the linear relation among three consecutive I_k's, viz.:

$$0 = kI_k^{i-1} - 2iI_k^i + kI_k^{i+1}.$$

In a paper which will be found in the *Lond. Math. Soc. Proc.*, Vol. IX., p. 61, Lord Rayleigh showed how these functions are related to Laplace's; but they will probably continue to be studied, as heretofore, from the point of view of a distinct system of transcendants, the theory of which Applied have contributed to Pure Mathematics. Their numerous applications to other parts of Physics than Astronomy are well-known. In the first volume of Tisserand's great work, *Mécanique Céleste*, of which the second volume has just appeared, they are early introduced.

That Laplace's Functions, about which so extensive a Pure-Mathematical literature has now gathered, were introduced in connexion with his researches on the attractions of spheroids[*] is too well known to be dwelt on.

Fourier's (1768–1830) Theorem, the applications of which have become so numerous, owing to its great generality, is one of the valuable additions which the Theory of Heat (1822) has contributed to the stock of Pure Mathematics, together with solutions of Partial Differential Equations. Of the many definite integrals which all branches of Natural Philosophy have brought under the notice of Pure Mathematicians, it would take much room to enumerate even a part;[†] while of Differential Equations De Morgan has well remarked[‡] that the extension of their theory is to be looked for in special works on gravitation, heat, &c.

In Sir R. Ball's "Cylindroid" (*Theory of Screws*, Ch. 2) is an instance of a geometrical surface brought into notice by dynamical researches, and which will for the future take its place in geometry as one of the cubical surfaces to be specially studied. So the "Cissoid" of Diocles(?) was the first plane cubic curve, and the "Conchoid" of Nicomedes (150 B.C.) the earliest plane quartic curve, studied in the search after a mechanical solution of the problems of the trisection of an angle and the determination of two mean proportionals.

The multipartition of an angle had in the fourth century B.C. led Dinostratus, pupil of Eudoxus, to the conception of the "Quadratrix."

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[*] *Méc. Céleste*, Tom. II., Ch. 2.

[†] As less known, Fresnel's Integrals may be particularly noticed.

[‡] *Diff. and Int. Calc.*, p. xiii., note.

Among the more recent forms of aid which Applied have offered to Pure Mathematics, is a method of proving theorems in the latter, of importance and great difficulty, from the consideration that the physical quantity represented by an analytical expression (as a definite integral) has an obvious concrete value, and thus the expression is at once evaluated. Among the instances of this method known to me are the late Professor Clerk-Maxwell's use of properties of the Potential to establish a theorem in Harmonic Analysis (*Elect. and Mag.*, Vol. I., § 135)—though this might hardly be considered a case in point—and Professor Crofton's remarkable evaluations of certain definite integrals through considerations of Local Probability (*Phil. Trans.*, 1868–9). Similarly, the consideration of a charged conductor has been used by Mr. Basset (*Camb. Phil. Soc. Proc.*, Vol. V., p. 426, and "Treatise on Hydrodynamics" II., p. 15), to prove that a function finite between given values of the variable can be expressed by means of a definite integral involving Bessel's Functions, taken between those values. This method of proof, to which, however, some mathematicians* have taken exception on abstract grounds, appears likely to be more frequently employed as writers on Pure Mathematics become more familiar with certain branches of physical research.

It may be proper, in conclusion, to give some account of those serial publications, quoted in the course of this essay, which began to appear in the second half of the sixteenth century as organs of communication among scientific, as well as literary men; and which may be regarded as the forerunners of the journals of the present time.

In 1665, the year preceding the foundation of the Académie Royale des Sciences, the *Journal des Sçavans*, written in Paris, but printed at Amsterdam, was commenced as a weekly issue to give accounts of new books, and to afford a medium for the discussion of all such topics as might interest the educated, whether in theology, metaphysics, law, medicine, chemistry, physics, mathematics or history. When greater freedom for the press obtained in France, the issue was transferred to Paris also, and, with short breaks, continued until 18th June, 1816. Recommenced in the following September as a monthly serial, it has been maintained without further interruption. The brothers Bernoulli were at first contributors to this Journal, which is referred to as *Ephemerides Gallicæ* and *Parisiense Diarium Eruditorum* in their Latin correspondence.

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* As Prof. Sylvester, through whose criticism (*Phil. Mag.*, 1876, II., p. 306) attention was directed to the first-cited instance.

In avowed imitation of the plan of the *Journal*, and in a similar form (called 12mo in the catalogues, but more of the size of a 24mo of the present time), the *Nouvelles de la République des Lettres* was commenced as a monthly publication, "to be written as well as printed" at Amsterdam, in March, 1684; and maintained, with breaks, until 1718. During one of these interruptions, a similar publication, with the title *Histoire des Ouvrages des Sçavans*, was started at Rotterdam, in September, 1687, and continued until June, 1709. The last two Journals are quoted sometimes as *Nova* and *Historia Bataviæ*, or *Nouvelles d'Amsterdam* and *de Rotterdam*, respectively.

The *Acta Eruditorum* (Note, p. 11), which was aided by a subvention from the Elector of Saxony, continued to appear in yearly volumes until 1777. The *Acta Helvetica*, quoted p. 15, was founded in 1751, and maintained until 1787. But, before the middle of the eighteenth century, the *Mémoires* of the Academies of Paris (1666) and Berlin (1702) had become the medium for the more elaborate essays of the analysts of the time.

Proofs of Steiner's Theorems relating to Circumscribed and Inscribed Conics. By Professor G. B. MATHEWS.

[*Read Nov. 13th*, 1890.]

The theorems here discussed are enunciated without proof in Steiner's memoir, entitled "Teoremi relativi alle coniche inscritte e circonscritte" (*Werke* II., p. 329, or *Crelle* XXX., p. 97). The most important are those relating to the maximum conics inscribed in a given quadrilateral, and the minimum conics circumscribed to a given quadrangle; the others, in fact, are preliminary to these, but, for the sake of completeness, proofs of them all have been given. It will be observed that the second principal problem admits of three proper solutions, besides nine improper ones, so that a purely geometrical method would necessarily involve the employment of curves other than conics and straight lines.

1. Adopting ordinary trilinear coordinates referred to a triangle ABC, let

$$\phi(x, y, z) \equiv (u, v, w, u', v', w' \between x, y, z)^2 = 0$$

be the equation of any conic, and let

$$\Phi(\xi, \eta, \zeta) \equiv (U, V, W, U', V', W' \between \xi, \eta, \zeta)^2 = 0$$

be the equation of the same conic in line-coordinates, so that

$$U = vw - u'^2, \text{ \&c.}$$

Further, let Δ denote the discriminant

$$\Delta = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

Then, if Γ be written for $\Phi(a, b, c)$, where a, b, c are the sides of ABC, the square of the area of the conic is equal to $M\Delta^2/\Gamma^3$, where M is a constant.

In order to determine M, let the conic be the circumcircle of ABC; then we may put

$$u = v = w = 0, \quad u' = a, \quad v' = b, \quad w' = c,$$

giving

$$\Delta = 2abc,$$

$$\Gamma = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$$

$$= 16S^2,$$

where S is the area of ABC.

Hence

$$\pi^2 R^4 = M \frac{a^2b^2c^2}{4(16)^3 S^6}$$

or, since

$$R = abc/4S,$$

$$M = 4\pi^2 a^2b^2c^2 S^2,$$

and the square of the area of the conic

$$= 4\pi^2 a^2b^2c^2 S^2 \Delta^2/\Gamma^3 \quad \text{..........................(1).}$$

2. Now, let

$$l^2x^2 + m^2y^2 + n^2z^2 - 2mnyz - 2nlzx - 2lmxy = 0$$

be any proper conic inscribed in ABC, and having its centre at (α, β, γ).

Then

$$l : m : n = a\,(b\beta+c\gamma-a\alpha) : b\,(c\gamma+a\alpha-b\beta) : (a\alpha+b\beta-c\gamma).$$

Suppose the sides of ABC are bisected in the points A', B', C'; then, if α', β', γ' are the perpendiculars from (α, β, γ), or P, upon the sides of $A'B'C'$,

$$\alpha' = S/a-\alpha = (b\beta+c\gamma-a\alpha)/2a, \text{ \&c.},$$

so that $$l : m : n = a^2\alpha' : b^2\beta' : c^2\gamma' \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For the conic now considered,

$$\Delta = -4l^2m^2n^2,$$

$$\Gamma = 4lmn\,(bcl+cam+abn)\,;$$

therefore, if E is the area of the conic,

$$E^2 = M\,\frac{lmn}{4\,(bcl+cam+abn)^3}$$

$$= M\,\frac{\alpha'\beta'\gamma'}{4abc\,(a\alpha'+b\beta'+c\gamma')^3};$$

whence, substituting for M its value, and observing that

$$a\alpha'+b\beta'+c\gamma' = S,$$

we find $$E^2 = 4\pi^2R\alpha'\beta'\gamma' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

3. In a similar way, if the circumscribed conic

$$2u'yz+2v'zx+2w'xy = 0$$

has its centre at (α, β, γ),

$$u' : v' : w' = a\alpha\alpha' : b\beta\beta' : c\gamma\gamma' \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and replacing these proportions by equalities, it will be found, after a few reductions, that

$$\Delta = 2abc\alpha\beta\gamma\alpha'\beta'\gamma',$$

$$\Gamma = 8Sabc\alpha'\beta'\gamma'.$$

Hence, if F be the area of the conic,

$$F^2 = \frac{\pi^2abc}{4S}\,\frac{\alpha^2\beta^2\gamma^2}{\alpha'\beta'\gamma'}$$

$$= \pi^2R\alpha^2\beta^2\gamma^2/\alpha'\beta'\gamma' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

4. Now, let conics with centre at P be inscribed and circumscribed to the triangle $A'B'C'$, and let $A''B''C''$ be derived from $A'B'C'$ in the same way as $A'B'C'$ from ABC. Then, if E_1, F_1 are the areas of the two new conics

$$E_1^2 = 4\pi^2 R_1 \alpha_1' \beta_1' \gamma_1',$$

$$F_1^2 = \pi^2 R_1 \alpha_1^2 \beta_1^2 \gamma_1^2 / \alpha_1' \beta_1' \gamma_1',$$

where $\quad R_1 = \tfrac{1}{2} R,$

$$\alpha_1, \beta_1, \gamma_1 = \alpha', \beta', \gamma',$$

$$\alpha_1', \beta_1', \gamma_1' = \tfrac{1}{2}(\alpha-\alpha'),\ \tfrac{1}{2}(\beta-\beta'),\ \tfrac{1}{2}(\gamma-\gamma');$$

so that $\quad E_1^2 = \tfrac{1}{4}\pi^2 R\,(\alpha-\alpha')(\beta-\beta')(\gamma-\gamma'),$

$$F_1^2 = 4\pi^2 R \alpha'^2 \beta'^2 \gamma'^2 / (\alpha-\alpha')(\beta-\beta')(\gamma-\gamma').$$

Hence, by multiplication,

$$E_1^2 F_1^2 = \pi^4 R^2 \alpha'^2 \beta'^2 \gamma'^2 = \tfrac{1}{16} E^4;$$

and therefore, taking the absolute values of E_1, F_1, that is, disregarding sign,

$$E^2 = 4E_1 F_1 \quad \text{..............................(6).}$$

The process of derivation may of course be repeated indefinitely.

But it is to be further observed that the conics E, F are such that an infinity of triangles can be drawn circumscribed to E, and inscribed in F. Let XYZ be any one of these, and let $X_0 Y_0 Z_0$ be related to it as ABC is to $A'B'C'$; then, if E_0 be the area of a conic with centre P inscribed in $X_0 Y_0 Z_0$, we have

$$E_0^2 = 4EF,$$

so that the area of all such conics E_0 is constant.

5. Suppose the inscribed conic E is constrained to touch the fixed line

$$\lambda x + \mu y + \nu z = 0;$$

then $\quad l\mu\nu + m\nu\lambda + n\lambda\mu = 0 \quad$ ....(7),

and therefore, by (2), the locus of P is the line

$$a^2\mu\nu\alpha' + b^2\nu\lambda\beta' + c^2\lambda\mu\gamma' = 0 \quad \text{......................(8).}$$

Conversely, if the centre of E describes the line

$$f\alpha'+g\beta'+h\gamma'=0,$$

E will always touch the fixed line

$$\frac{a^2}{f}x+\frac{b^2}{g}y+\frac{c^2}{h}z=0.$$

The triangle of reference and the line

$$\lambda x+\mu y+\nu z=0$$

form a quadrilateral in which an infinity of conics, as above, can be inscribed. In order to find the conics of maximum area, we have to make

$$\alpha'\beta'\gamma'$$

a maximum, subject to the conditions

$$a^2\mu\nu\alpha'+b^2\nu\lambda\beta'+c^2\lambda\mu\gamma'=0 \text{ ....}(9),$$

$$a\alpha'+b\beta'+c\gamma'=S \text{ ....}(10).$$

The ordinary process leads to the equation

$$\begin{vmatrix} \beta'\gamma', & \gamma'\alpha', & \alpha'\beta' \\ a^2\mu\nu, & b^2\nu\lambda, & c^2\lambda\mu \\ a, & b, & c \end{vmatrix}=0 \text{ ....}(11),$$

and the centres of the maximum conics are determined as the intersections of the line (9) with the conic (11). The latter conic goes through A', B', C', through the centroid, G, of ABC (or $A'B'C'$), and through the point for which

$$a^2\alpha' : b^2\beta' : c^2\gamma' = \lambda : \mu : \nu;$$

that is, by (2), through the centre of

$$\lambda^2x^2+\mu^2y^2+\nu^2z^2-2\mu\nu yz-2\nu\lambda zx-2\lambda\mu xy=0.$$

This point can be easily found geometrically. Namely, if

$$\lambda x+\mu y+\nu z=0$$

meet the sides of ABC in D, E, F, and the points D', E', F' be taken

so that

$$(BCDD') = (CAEE') = (ABFF') = -1,$$

then the lines joining A, B, C to the middle points of $E'F'$, $F'D'$, $D'E'$, respectively, will meet together at the required point O.

Since five points upon it are known, the conic can be constructed geometrically; the line represented by (9) is, as is well known, the line through the middle points of the diagonals of the quadrilateral $BCEF$. This being drawn, its intersections with the conic $OA'B'C'G$ are the centres of the maximum conics.

6. From a geometrical point of view, the construction above indicated is about as simple and satisfactory as could be expected: it is interesting, however, to verify a remarkable metrical theorem of Steiner's with reference to the centres, K, K' suppose, of the maximum conics. The theorem is that, if the line of centres meet the sides of $A'B'C'$ in P_1, P_2, P_3, and if M bisect KK', then M is the mean centre of P_1, P_2, P_3, and

$$MK^2 = MK'^2 = \tfrac{1}{6}\,(MP_1^2+MP_2^2+MP_3^2)\,;$$

also, M is the centroid of the six vertices of the given circumscribing quadrilateral.

In order to verify these statements, we may solve the equations (9), (10), (11). It will simplify the result if we write p, q, r for the determinants of the matrix

$$\begin{vmatrix} a, & b, & c \\ \lambda, & \mu, & \nu \end{vmatrix},$$

so that $$p = b\nu-c\mu, \quad q = c\lambda-a\nu, \quad r = a\mu-b\lambda \;\ldots\ldots\ldots\ldots(12).$$

Further, let us put

$$\begin{aligned} Z &= b^2c^2\lambda^2+c^2a^2\mu^2+a^2b^2\nu^2-abc\,(a\mu\nu+b\nu\lambda+c\lambda\mu) \\ &= \tfrac{1}{2}\,(a^2p^2+b^2q^2+c^2r^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13), \end{aligned}$$

an essentially positive quantity.

Then the coordinates of K, K', referred to the triangle $A'B'C'$, are explicitly given by

$$\left.\begin{aligned} \alpha &= \frac{\lambda S}{3aqr}\,(cr-bq\pm\sqrt{Z}) \\ \beta &= \frac{\mu S}{3brp}\,(ap-cr\pm\sqrt{Z}) \\ \gamma &= \frac{\nu S}{3cpq}\,(bq-ap\pm\sqrt{Z}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

The coordinates of M, referred to the same triangle, are therefore

$$\alpha_0 = \frac{\lambda S}{3aqr}\,(cr-bq),\quad \beta_0 = \frac{\mu S}{3brp}\,(ap-cr),\quad \gamma_0 = \frac{\nu S}{3cpq}\,(bq-ap)\ldots(15),$$

while those of P_1, P_2, P_3 are respectively

$$\left.\begin{aligned} \alpha_1,\ \beta_1,\ \gamma_1 &= 0,\quad -c\mu S/bp,\quad b\nu S/cp \\ \alpha_2,\ \beta_2,\ \gamma_2 &= c\lambda S/aq,\quad 0,\quad -a\nu S/cq \\ \alpha_3,\ \beta_3,\ \gamma_3 &= -b\lambda S/ar,\quad a\mu S/br,\quad 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(16).$$

Hence we find without difficulty

$$\alpha_0 = \tfrac{1}{3}\,(\alpha_1+\alpha_2+\alpha_3),\ \&c.,$$

that is, M is the mean centre of P_1, P_2, P_3.

$$\begin{aligned} \text{Also,}\quad & (\alpha_0-\alpha_1)^2+(\alpha_0-\alpha_2)^2+(\alpha_0-\alpha_3)^2 \\ &= \alpha_1^2+\alpha_2^2+\alpha_3^2-3\alpha_0^2 \\ &= \lambda^2S^2\left\{\frac{c^2}{a^2q^2}+\frac{b^2}{a^2r^2}-\frac{(cr-bq)^2}{3a^2q^2r^2}\right\} \\ &= \frac{2\lambda^2S^2}{3a^2q^2r^2}\,(c^2r^2+b^2q^2+bcqr) \\ &= \frac{\lambda^2S^2}{3a^2q^2r^2}\,\{b^2q^2+c^2r^2+(bq+cr)^2\} \\ &= \frac{\lambda^2S^2}{3a^2q^2r^2}\,(a^2p^2+b^2q^2+c^2r^2) \\ &= \frac{2\lambda^2S^2Z}{3a^2q^2r^2} = \frac{3}{2}\left(\frac{2\lambda S}{3aqr}\sqrt{Z}\right)^2. \end{aligned}$$

From this, the two similar equations, and (14), it follows that

$$MP_1^2+MP_2^2+MP_3^2 = \tfrac{3}{2}KK'^2 = 6MK^2.$$

7. It is possible that the foregoing method, or something like it, is that by which Steiner obtained his results, and that he refrained from publishing a proof of this kind, because he hoped to obtain one of a more purely geometrical character. If, however, we try to solve in a similar manner the analogous problem of finding conics of minimum area which pass through four given points, we are led to results of great complexity. It will be found that the centres of the conics of stationary area are determined as the intersections of a conic and a sextic, so that apparently there are twelve solutions; it will appear, however, from a less symmetrical, but more manageable method, to be presently explained, that there are only three proper solutions, excluding the line-pairs, and the parabolas of the system; whence we infer that the set of twelve solutions is made up by the three proper solutions, the three line-pairs, and the two parabolas each reckoned three times, the conic which is the locus of centres in fact osculating the sextic where it meets it at infinity.

8. Even the problem already discussed may be more simply treated by the unsymmetrical method.

Taking line coordinates ξ, η, ζ, the conic

$$(1+t)\,\lambda\eta\zeta-\mu\zeta\xi-t\nu\xi\eta = 0 \quad\dots\dots\dots\dots\dots\dots(17),$$

where t is a variable parameter, touches the three sides of the triangle of reference, and the fixed line (λ, μ, ν).

As in Art. 1, it can be shown that the square of the area of the conic is proportional to

$$t\,(1+t)\,/\,(bqt-cr)^3.$$

For a stationary value, we have, by logarithmic differentiation,

$$\frac{1}{t}+\frac{1}{t+1}-\frac{3bq}{bqt-cr} = 0 \text{ or } \infty.$$

It is easily seen that

$$t = 0, \quad -1, \quad \infty$$

give improper solutions, namely, the point-pairs of the system; while

$$bqt-cr = 0,$$

corresponds to the single parabola of the system.

The remaining (proper) solutions are given by

$$bqt^2+2\,(bq+cr)\,t+cr = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

or

$$bqt^2-2apt+cr = 0.$$

Eliminating t between (17) and (18), the equation of the maximum conics is obtained in the form

$$ap\lambda^2\eta^2\zeta^2+bq\mu^2\zeta^2\xi^2+cr\nu^2\xi^2\eta^2+2\xi\eta\zeta\,(ap\mu\nu\xi+bq\nu\lambda\eta+cr\lambda\mu\zeta) = 0\ldots(19).$$

9. Actually solving (18), we get, in our previous notation,

$$t = \frac{ap\pm\sqrt{Z}}{bq} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

and equation (17) becomes

$$(cr\mp\sqrt{Z})\,\lambda\eta\zeta+bq\mu\zeta\xi+(ap\pm\sqrt{Z})\,\xi\eta = 0.$$

Now, with the help of the identity

$$ap+bq+cr = 0,$$

it can be verified that

$$\left.\begin{aligned}(cr+\sqrt{Z})(ap-cr+\sqrt{Z}) &= ap\,(cr-bq+\sqrt{Z})\\ (ap-\sqrt{Z})(ap-cr+\sqrt{Z}) &= cr\,(bq-ap+\sqrt{Z})\end{aligned}\right\} \;\ldots\ldots\ldots(21),$$

so that the equations of the maximum conics are separately obtained in the symmetrical form

$$ap\,(cr-bq\pm\sqrt{Z})\,\lambda\eta\zeta+bq\,(ap-cr\pm\sqrt{Z})\,\mu\zeta\xi$$
$$+cr\,(bq-ap\pm\sqrt{Z})\,\nu\xi\eta = 0\ldots\ldots\ldots(22).$$

It is now easy to obtain the coordinates of their centres, &c., and thus to verify the results of Art. 6.

10. Proceeding now to the other problem, we have, in point-coordinates,

$$(1+t)\,fyz-gzx-thxy = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(23),$$

representing a conic through the vertices of the triangle of reference, and through the fixed point (f, g, h). In order that its area may be a maximum or minimum, the value of

$$\frac{t^2(1+t)^2}{\{(af+ch)^2t^2+2(a^2f^2-bcgh+cahf+abfg)\,t+(af+bg)^2\}^3}$$

must be stationary.

Representing the denominator by u, for the moment, we have

$$\frac{2}{t}+\frac{2}{t+1}-\frac{3}{u}\cdot\frac{du}{dt}=0 \text{ or } \infty.$$

As before, $t=0, -1, \infty$ give the three line-pairs of the system, while $u=0$ corresponds to the two parabolas of the system. Rejecting these solutions as irrelevant, we obtain a cubic equation in t, which, written out in full, is

$$(af+ch)^2t^3+(a^2f^2+2c^2h^2+bcgh+3cahf-abfg)\,t^2$$
$$-(a^2f^2+2b^2g^2+bcgh-cahf+3abfg)\,t-(af+bg)^2=0\ldots(24).$$

Eliminating t between (23) and (24), we obtain the equation of the three critical conics in the following form, where, for simplicity, X, Y, Z are written for fyz, gzx, hxy respectively :—

$$(bg+ch)^2X^3+(ch+af)^2Y^3+(af+bg)^2Z^3$$
$$-(2b^2g^2+c^2h^2+3bcgh-cahf+abfg)\,X^2Y$$
$$-(b^2g^2+2c^2h^2+3bcgh+cahf-abfg)\,X^2Z$$
$$-(2c^2h^2+a^2f^2+3cahf-abfg+bcgh)\,Y^2Z$$
$$-(c^2h^2+2a^2f^2+3cahf+abfg-bcgh)\,Y^2X$$
$$-(2a^2f^2+b^2g^2+3abfg-bcgh+cahf)\,Z^2X$$
$$-(a^2f^2+2b^2g^2+3abfg+bcgh-cahf)\,Z^2Y$$
$$+4(a^2f^2+b^2g^2+c^2h^2+bcgh+cahf+abfg)\,XYZ=0\ \ldots(25).$$

The separate equations of the conics are to be obtained, either by solving (24) and substituting in (23), or by applying the theory of ternary cubics to resolve the left-hand side of (25) into three factors, each linear in X, Y, Z.

On an Algebraic Integral of Two Differential Equations.

By R. A. Roberts.

[*Read Nov.* 13*th*, 1890.]

1. If u, v denote two quadratic expressions in a variable x, I propose to show that the differential equations

$$\frac{dx_1}{\sqrt[3]{(u_1v_1^2)}}+\frac{dx_2}{\sqrt[3]{(u_2v_2^2)}}+\frac{dx_3}{\sqrt[3]{(u_3v_3^2)}}=0,$$

$$\frac{dx_1}{\sqrt[3]{(u_1^2v_1)}}+\frac{dx_2}{\sqrt[3]{(u_2^2v_2)}}+\frac{dx_3}{\sqrt[3]{(u_3^2v_3)}}=0,$$

where x_1, x_2, x_3 are three values of x, and u_1, v_1; u_2, v_2; u_3, v_3 the corresponding values of u, v, necessarily involve an algebraic relation between the variables x_1, x_2, x_3 containing two arbitrary constants.

2. Let

$$u = lx^2+mx+n, \quad v = l'x^2+m'x+n';$$

then, if

$$f(x) = (\alpha+\beta x)^3u-(\gamma+\delta x)^3v \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

there are evidently two relations, independent of α, β, γ, δ, connecting the roots of the quintic

$$f(x) = 0.$$

These may be easily found in the form of systems of determinants involving the cube roots of u, v; for, if x_1, x_2, x_3, x_4 are four roots of

$$f(x) = 0,$$

we have

$$(\alpha+\beta x_1)\sqrt[3]{u_1}-(\gamma+\delta x)\sqrt[3]{v_1} = 0,$$

and three other similar equations, from which we get, eliminating α, β, γ, δ,

$$\begin{vmatrix} \sqrt[3]{u_1}, & x_1\sqrt[3]{u_1}, & \sqrt[3]{v_1}, & x_1\sqrt[3]{v_1} \\ \sqrt[3]{u_2}, & x_2\sqrt[3]{u_2}, & \sqrt[3]{v_2}, & x_2\sqrt[3]{v_2} \\ \sqrt[3]{u_3}, & x_3\sqrt[3]{u_3}, & \sqrt[3]{v_3}, & x_3\sqrt[3]{v_3} \\ \sqrt[3]{u_4}, & x_4\sqrt[3]{u_4}, & \sqrt[3]{v_4}, & x_4\sqrt[3]{v_4} \end{vmatrix} = 0 \;\ldots\ldots\ldots\ldots\; (2);$$

and, in the same way, we have four other determinants involving x_1, x_2, x_3, x_4 and the fifth root x_5.

3. We may also find these relations in other forms, for, substituting a, b, the roots of $u=0$, and c, d, the roots of $v=0$, in the identity (1) successively, we get

$$\left.\begin{array}{ll}\sqrt[3]{f(a)} \propto \gamma+\delta a, & \sqrt[3]{f(b)} \propto \gamma+\delta b \\ \sqrt[3]{f(c)} \propto \alpha+\beta c, & \sqrt[3]{f(d)} \propto \gamma+\delta d\end{array}\right\} \text{..............} (3);$$

from which and the relations

$$(\alpha+\beta x_1)\sqrt[3]{u_1}-(\gamma+\delta x_1)\sqrt[3]{v_1}=0,$$

&c., we can eliminate α, β, γ, δ linearly, and so obtain a number of other determinants. A particular form of the two relations in this case, when two of the roots of

$$f(x)=0$$

are supposed to be given, is worth noticing. From

$$(\alpha+\beta x_4)\sqrt[3]{u_1}-(\gamma+\delta x_4)\sqrt[3]{u_4}=0,$$

$$(\alpha+\beta x_5)\sqrt[3]{u_5}-(\gamma+\delta x_5)\sqrt[3]{u_5}=0,$$

where x_4, x_5 are the given roots, we have two linear relations connecting α, β, γ, δ, so that from (3) we obtain

$$\left.\begin{array}{l}\lambda\sqrt[3]{f(a)}+\mu\sqrt[3]{f(b)}+\nu\sqrt[3]{f(c)}=0 \\ \lambda'\sqrt[3]{f(a)}+\mu'\sqrt[3]{f(b)}+\nu'\sqrt[3]{f(d)}=0\end{array}\right\} \text{..............} (4),$$

where λ, μ, ν, λ', μ', ν' are known quantities, and $f(a)$, $f(b)$, $f(c)$, $f(d)$ are respectively proportional to

$$(a-x_1)(a-x_2)(a-x_3),\quad (b-x_1)(b-x_2)(b-x_3),$$

$$(c-x_1)(c-x_2)(c-x_3),\quad (d-x_1)(d-x_2)(d-x_3).$$

4. I now proceed to show that the relations between the five roots of

$$f(x)=0$$

can be written in a form involving differentials.

Suppose α, β, γ, δ to be functions of a variable t; then, if

$$f(x)=0$$

is a relation connecting the variables x and t, we obtain, by differentiation,

$$f'(x)\frac{dx}{dt}+3(\alpha+\beta x)^2(\alpha'+\beta' x)u-3(\gamma+\delta x)^2(\gamma'+\delta' x)v=0 \text{......} (5),$$

where α', β', γ', δ' are the differential coefficients of α, β, γ, δ with respect to t. Dividing out now by $\sqrt[3]{(uv^2)}f'(x)$, we get

$$\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\frac{3(\alpha+\beta x)^2(\alpha'+\beta' x)}{f'(x)}\sqrt[3]{\left(\frac{u^2}{v^2}\right)}$$

$$-\frac{3(\gamma+\delta x)^2(\gamma'+\delta' x)}{f'(x)}\sqrt[3]{\left(\frac{v}{u}\right)}=0,$$

which, since

$$(\alpha+\beta x)^3 u=(\gamma+\delta x)^3 v,$$

becomes

$$\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\frac{3(\gamma+\delta x)^2(\alpha'+\beta' x)-3(\gamma+\delta x)(\alpha+\beta x)(\gamma'+\delta' x)}{f'(x)}=0 \text{..................} (6).$$

Now, let x take successively the values of the five roots of the equation

$$f(x)=0;$$

then, summing with regard to these five quantities, we obtain

$$\Sigma\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\Sigma\frac{\phi(x)}{f'(x)}=0,$$

where $\phi(x)$ is an expression of the third degree in x; but the expression on the right-hand side, namely

$$\Sigma\frac{\phi(x)}{f'(x)},$$

vanishes, in accordance with a well-known theorem. Hence we have

$$\Sigma\frac{dx}{\sqrt[3]{(uv^2)}}=0 \text{..............................} (7),$$

omitting the variable t.

In precisely the same way, by dividing (5) by $\sqrt[3]{(u^2v)}f'(x)$ and summing, we get

$$\Sigma\frac{dx}{\sqrt[3]{(u^2v)}}=0 \text{..............................} (8).$$

5. If we suppose now that two of the roots x_4, x_5 of $f(x)$ are constants, these quantities will be eliminated in the differential equations,

as there result

$$\frac{dx_1}{\sqrt[3]{(u_1v_1^2)}}+\frac{dx_2}{\sqrt[3]{(u_2v_2^2)}}+\frac{dx_3}{\sqrt[3]{(u_3v_3^2)}}=0,$$

$$\frac{dx_1}{\sqrt[3]{(u_1^2v_1)}}+\frac{dx_2}{\sqrt[3]{(u_2^2v_2)}}+\frac{dx_3}{\sqrt[3]{(u_3^2v_3)}}=0,$$

which are the equations that we proposed to show were equivalent to two algebraic conditions connecting x_1, x_3. These latter may be most simply written in the forms (4), namely,

$$\lambda^3 f(a)+\mu^3 f(b)+\nu\sqrt[3]{f}(c)=0,$$

$$\lambda'\sqrt[3]{f}(a)+\mu'\sqrt[3]{}(b)+\nu\sqrt[3]{f}(d)=0.$$

6. The foregoing results are a special application of the general theorem of Abel concerning the comparison of transcendents; but seem worth noticing on account of their simplicity. The algebraical relations are thus consistent with the transcendental equations, in consequence of the integrals belonging to the class called Abelian. It seems, however, worth giving some special attention to the integrals involved.

Let J denote the Jacobian quadratic of u, v; then we have

$$udv-vdu=Jdx;$$

therefore
$$\int\frac{dx}{\sqrt[3]{(uv^2)}}=\int\frac{udv-vdu}{J\sqrt[3]{(uv^2)}};$$

but by a known relation, we have

$$J^2=\alpha u^2+\gamma uv+\beta v^2,$$

where α, β, γ are functions of the coefficients of u, v, and, in fact, α, β are proportional to the discriminants of v, u, respectively.

Hence
$$\int\frac{dx}{\sqrt[3]{(uv^2)}}=\int\frac{udv-vdu}{\sqrt[3]{(uv^2)}\sqrt{(\alpha u^2+\gamma uv+\beta v^2)}},$$

which becomes
$$3\int\frac{dz}{\sqrt{(\alpha+\gamma z^3+\beta z^6)}}\quad\text{..........................}\quad(9),$$

by putting
$$v=uz^3.$$

In the same way

$$\int\frac{dx}{\sqrt[3]{(u^2v)}}=3\int\frac{zdz}{\sqrt{(\alpha+\beta z^6+\gamma z^3)}}\quad\text{......................}\quad(10).$$

Now, these integrals in z are apparently of the first class of hyper-elliptic integrals, but can be reduced to elliptic integrals, which are, however, not necessarily real, as I proceed to prove.

7. First, suppose that α and β have the same sign, that is, that the discriminants of u, v have the same sign, in which case u, v have their factors both real or both imaginary; then, writing

$$\alpha = \beta k^6,$$

the foregoing integrals become

$$\int \frac{dz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^3\}}}, \quad \int \frac{z\,dz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^3\}}}.$$

Putting now
$$z + \frac{k^2}{z} = y,$$

we have
$$z^3 + \frac{k^6}{z^3} = y^3 - 3k^2 y,$$

and
$$2\sqrt{z} = \sqrt{(y+2k)} \pm \sqrt{(y-2k)},$$

$$\frac{2k}{\sqrt{z}} = \sqrt{(y+2k)} \mp \sqrt{(y-2k)},$$

so that
$$\frac{2\,dz}{\sqrt{z}} = \frac{dy}{\sqrt{(y-2k)}} \pm \frac{dy}{\sqrt{(y+2k)}},$$

$$\frac{2k\,dz}{\sqrt{z^3}} = \frac{dy}{\sqrt{(y+2k)}} \mp \frac{dy}{\sqrt{(y-2k)}}.$$

Hence

$$\int \frac{dz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^6\}}} = \int \frac{dz}{\sqrt{z^3}} \frac{1}{\sqrt{\left\{\beta\left(z^3+\frac{k^6}{z^3}\right)+\gamma\right\}}}$$

$$= \frac{1}{2k} \int \left\{ \frac{dy}{\sqrt{(y-2k)}} \mp \frac{dy}{\sqrt{(y+2k)}} \right\} \frac{1}{\sqrt{\{\beta(y^3-3k^2y)+\gamma\}}},$$

and
$$\int \frac{z\,dz}{\sqrt{\{\alpha(z^6+k^6)+\gamma z^3\}}} = \int \frac{dz}{\sqrt{z}\sqrt{\left\{\beta\left(z^3+\frac{k^6}{z^3}\right)+\gamma\right\}}}$$

$$= \frac{1}{2} \int \left\{ \frac{dy}{\sqrt{(y+2k)}} \pm \frac{dy}{\sqrt{(y-2k)}} \right\} \frac{1}{\sqrt{\{\beta(y^3-3k^2y)+\gamma\}}}.$$

We see thus that (9) and (10) are made to depend upon the elliptic integrals

$$\int \frac{dy}{\sqrt{\{(y+2k)(\beta y^3-3k^2\beta y+\gamma)\}}},$$

$$\int \frac{dy}{\sqrt{\{(y-2k)(\beta y^3-3k^2\beta y+\gamma)\}}},$$

which are real if, as we have supposed, a, β have the same signs; but if the latter have not, that is, if u, say, has real factors, and v imaginary ones, then k is imaginary, and the elliptic integrals are also, and cannot be resolved into their real and imaginary parts, except by means of the more general hyper-elliptic integrals from which they were derived.

8. Some integrals which come under the preceding forms may be noticed. If

$$u = x, \quad v = ax^2+bx+c,$$

and we then put $x = z^3$, we have

$$\int \frac{dx}{\sqrt[3]{(u^2v)}} = 3\int \frac{dz}{(az^6+bz^3+c)^{\frac{1}{3}}},$$

$$\int \frac{dx}{\sqrt[3]{(uv^2)}} = 3\int \frac{zdz}{(az^6+bz^3+c)^{\frac{2}{3}}}.$$

Thus, from what we have proved, the integrals on the right-hand side can be expressed by means of elliptic integrals.

9. An application of the preceding results may be made, so as to obtain the differential equations of a certain system of lines in space satisfying two conditions.

Let x, y, z, u be quadriplanar coordinates of a point; then, if we have the system of cubics

$$\frac{x^3}{a-\lambda}+\frac{y^3}{b-\lambda}+\frac{z^3}{c-\lambda}+\frac{u^3}{d-\lambda} = 0,$$

the coordinates of any point in space can be expressed in terms of the parameters λ_1, λ_2, λ_3 of the three cubics of the system which pass through the point, as follows:

$$x^3 = \frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(a-b)(a-c)(a-d)}, \quad y^3 = \frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{(b-a)(b-c)(b-d)},$$

$$z^3 = \frac{(c-\lambda_1)(c-\lambda_2)(c-\lambda_3)}{(c-a)(c-b)(c-d)}, \quad u^3 = \frac{(d-\lambda_1)(d-\lambda_2)(d-\lambda_3)}{(d-a)(d-b)(d-c)};$$

so that, if

$$\frac{d\lambda_1}{\sqrt[3]{(u_1v_1^2)}}+\frac{d\lambda_2}{\sqrt[3]{(u_2v_2^2)}}+\frac{d\lambda_3}{\sqrt[3]{(u_3v_3^2)}}=0,$$

$$\frac{d\lambda_1}{\sqrt[3]{(u_1^2v_1)}}+\frac{d\lambda_2}{\sqrt[3]{(u_2^2v_2)}}+\frac{d\lambda_3}{\sqrt[3]{(u_3^2v_3)}}=0,$$

where $u=(a-\lambda)(b-\lambda),\quad v=(c-\lambda)(d-\lambda),$

the integrals (4) will give

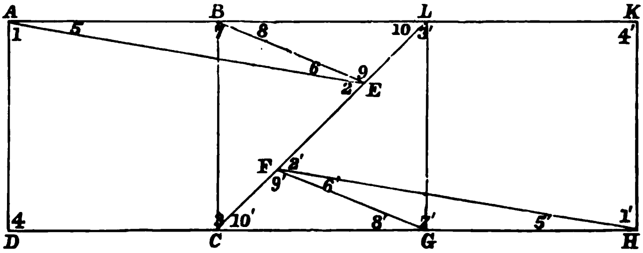
$$lx+my+nz=0,\quad l'x+m'y+n'u=0,$$

where l, m, n, l', m', n' are quantities involving the two constants introduced by integration; that is, the differential equations represent a system of lines in space.

Some Theorems in Elementary Geometry. *By* Mr. Oscher Ber.

[*Read Nov.* 13*th*, 1890.]

I. *To describe a square which is equal to three given equal squares.* Place the squares side by side as in the figure.



Take E on CL so that $CE=CB$,

,, F ,, CL ,, $LF=CB$.

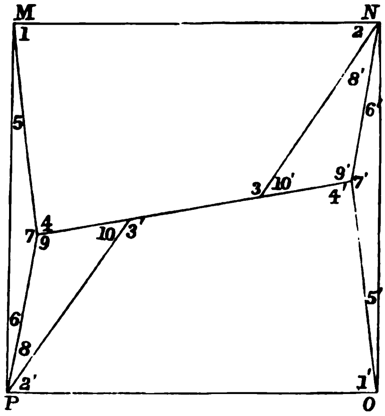
Then cut out the figures

$AECD$, $HKLF$, AEB, HFG, BEL, CFG.

The angles have all been numbered.

The angles 1, 1′ are equal; 2, 2′ are equal, and so on.

The parts can now be arranged thus:—



The angles 3, 10′ make up two right angles, and so do 3′, 10.

The angles 4, 7, 9,—*i.e.*, ADC, ABE, BEL,—together equal

$$ADC+ABC+CBE+CBE+ECB.$$

Now $CB = CE$;

therefore $CBE = CEB$;

therefore 4, 7, 9 together $= ADC+ABC+CBE+CEB+ECB$

$= 4$ right angles.

The side 5 7 = side 1 4.

The side 6 7 = side 8 9.

The angles 6, 8, 2′ together $= BEA+EBL+LFH$

$= BEA+EBL+AEC$

$= BEC+EBL$

$= CBE+EBL$

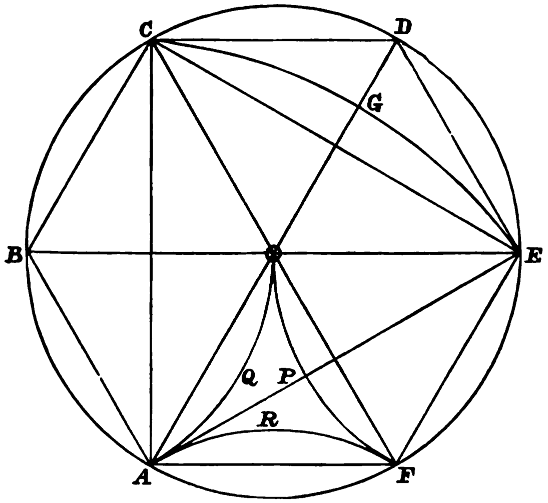
$= CBL =$ a right angle.

The angles 1, 5 together = 1′, 5′ together = a right angle.

The sides 1 2 and 5 6 are equal.

Hence the figure $MNOP$ is a square equal to three given equal squares.

2. *On an area equal to a given semicircle.*



Inscribe a regular hexagon $ABCDEF$ in the circle $ABCDEF$.

With A as centre, and AC as radius, describe a circular arc CE.

Then the figure bounded by the straight lines AC, AE, and the arc CGE is equal to half the whole circle.

For the triangle $AEC =$ half the hexagon.

It remains, therefore, to show that the segment $CGEC$ is half the sum of the six segments of the circle outside the hexagon.

Consider the segments $CGEC$ and the segment of the circle cut off by DE.

The arc of the segment $CGEC$ subtends at the centre A of its circle an angle equal to the angle of an equilateral triangle.

The arc of the segment DE subtends at centre O of its circle the same angle.

Hence the segments are similar.

Hence area of segment $CEGC$: area of segment cut off by DE

$$:: CE^2 : DE^2 :: 3 : 1;$$

therefore area of segment $CEGC$

$$= 3 \text{ (area of segment cut off by } DE)$$
$$= \tfrac{1}{2} \text{ sum of 6 segments outside hexagon;}$$

but triangle $AEC = \frac{1}{2}$ of hexagon;

therefore figure bounded by straight lines AE, AC, and arc CGE
$= \frac{1}{2}$ of whole circle.

3. If with E as centre and EO as radius the arc OPF be drawn,
if with B ,, ,, BO ,, ,, OQA ,,
and if an equal arc be described on AF:

Then these three circular arcs touch where they meet, and form a triangular figure $OQARFPO$.

Now, each of the segments $OQAO$, $ARFA$, $OPFO$ is equal to each of the segments outside the hexagon.

Therefore their sum is equal to the segment $CGEC$.

Also triangle $AOF =$ triangle CED.

Hence the figure $OQARFPO$ (bounded by three arcs and triangular) $=$ figure $CDEGC$ (bounded by two straight lines and one arc).

On the Analytical Representation of Heptagrams.

By L. J. ROGERS.

[*Read November 13th,* 1890.]

CONTENTS.

1. Hermite's Conditions that ϕx should represent a Substitution.
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1. In the standard works on the Theory of Substitutions—I allude to Jordan's *Traité des Substitutions*, Netto's *Substitutionentheorie*, and to those sections in Serret's *Algebra* referring to the same—there is an extract made from a paper of M. Hermite's, which appeared in the

Comptes Rendus, Vol. LVII., on the method of analytically expressing a substitution of a prime number of letters by means of a congruence-quantic of order two unities less than the prime in question. Since the subject is dealt with in greatest detail by Serret, it will be best to make all references to his work in preference to the other works mentioned above.

To explain this more fully, let us take p symbols, which we may adequately represent by the numbers $0, 1, 2, 3 \ldots (p-1)$, and suppose that p is prime. Now, suppose these symbols subjected to a substitution so that the arrangement becomes shifted into $\alpha, \beta, \gamma \ldots$, which are the same symbols in a different order. Then it is proved in Serret's *Algebra*, § 474, that an algebraic function ϕx can always be found, such that

$$\phi 0 \equiv \alpha, \text{ mod. } p, \quad \phi 1 \equiv \beta, \quad \phi 2 \equiv \gamma, \text{ \&c.},$$

for the whole set of symbols, and this function can by Fermat's theorem always be reduced to degree $(p-2)$, at the highest.

For instance, if $p = 7$, and the substitution be that of rearranging 0, 1, 2, 3, 4, 5, 6 into 2, 4, 1, 5, 6, 3, 0, we shall find that

$$\phi x \equiv 3x^5 + 4x^3 + 2x + 2.$$

However, the converse is not always true, that every quantic of order not greater than $(p-2)$ should represent a substitution. For, since we must have the final set of symbols identical with the first, though not in the same order, it is obvious that the only conditions we must and need but have, are that $\phi 0$, $\phi 1$, $\phi 2$, &c. should be congruent non-respectively with 0, 1, 2, 3 ... &c., or, in other words, all different. Thus, for modulus 7, the function $x^3 + x^2$ does not represent a substitution, for its values got by giving x the values 0, 1, 2... are 0, 2, 5, 1, 3, 3, 0, which are not all different.

Now, in general, it is obvious from this consideration, that if ϕx represents a substitution, so also will $\phi x + k$, where k is independent of x. Hence, in testing these forms, it is sufficient to test those in which

$$\phi 0 \equiv 0.$$

The necessary and sufficient conditions have been discovered by M. Hermite, and put into the simple and elegant form, that the $(p-3)$ coefficients of x^{p-1}, obtained in calculating the values of the powers of ϕx, viz.:

$$(\phi x)^2, \; (\phi x)^3 \ldots (\phi x)^{p-2},$$

and reducing by Fermat's theorem all powers greater than the $(p-1)^{\text{th}}$, should be severally congruent with zero (see § 476).

As I intend to deal in detail with quantics representing substitutions of seven letters, I shall now leave the general case, and take all congruences according to the modulus 7.

It is also my object to refer especially to the geometrical significance of these quantics, and to show how a substitution of seven letters may be adequately represented by a seven-point polygon or heptagram, understood in its most general form; either as a complete heptagon, or as a triangle and a quadrangle, or as a pentagon and a line, &c. This may be done by arranging seven dots, exactly or approximately at the vertices of a perfectly regular heptagon, and numbering them 0, 1, 2, 3, 4, 5, 6, in order. We will agree, moreover, to place the zero-vertex vertically highest, so to speak, and arrange the others symmetrically about the *vertical* line through the zero-vertex, ascending in value in the direction of the hands of a clock.

If then $$\phi\alpha \equiv \beta$$

for any particular values α, β, we shall join the vertex α to the vertex β, and when all vertices are thus joined, we shall get a complete seven-point figure. With this convention, it will not always be necessary to number the vertices.

For instance, if

$$\phi x \equiv 3x^5 + 4x^3 + 2x + 2,$$

we shall easily trace the figure from the values of $\phi 0$, $\phi 1$, &c., given above, namely, 2, 4, 1, 5, 6, 3, 0, to be that given in Fig. 1, while the arrow-head marks the direction in which we draw the lines. This is obviously necessary for each sub-polygon, except in the case of a single line or dot. Here we need only place an arrow-head on one side of the pentagon.

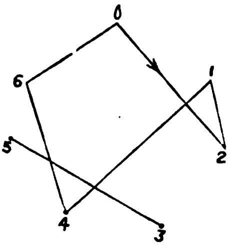

Fig. 1.

M. Hermite has found that all substitutions of 7 letters may be represented by the functions

$$az + \beta, \quad \text{or} \quad a\theta(z+\beta) + \gamma,$$

where $$\theta z \equiv z^4 \pm 3z \quad \text{.................................} (1),$$

or $$z^5 \pm 2z^2 \quad \text{.................................} (2),$$

or $$z^5+az^3+3a^2z, \text{ where } a^7-a\equiv 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

or $$z^5+az^3\pm z^2+3a^2z, \text{ where } a^3\equiv -1 \quad\ldots\ldots\ldots\ldots\ldots(4);$$

giving in all 5040 forms, as is shown in Serret's *Algebra*. These simplified forms he refers to as *formes réduites*, or reducts.

2. I propose, however, to rearrange these in seven classes, on the principle that, if θz be any form, then every reduct of the same form may be congruent with $\lambda\theta\mu z$, where λ, μ are constant. This will necessitate dividing $z^5+az^3+3a^2z$ into three new classes, according as

$$a\equiv 0, \quad a^3\equiv 1, \quad \text{or} \quad a^3\equiv -1.$$

If we give θz the following values—

$$z^4+3m^3z, \quad z^5+2m^3z^2, \quad z^5, \quad z^5+m^2z^3+3m^4z, \quad z^5-m^2z^3+3m^4z,$$
$$z^5+3m^2z^3+m^3z^2+6m^4z,$$

we shall obtain, by generalizing, the same 5040 forms as obtained by M. Hermite.

Now, these reducts may be replaced by others, equally general, and of such forms that M. Hermite's conditions may be immediately verified.

Thus, it can be easily shown that

$$z^5+m^2z^3+3m^4z\equiv 5m^4z\,(z^2+m^2)^3,$$

and that $$z^5-m^2z^3+3m^4z\equiv 4z^5\,(z^4+\lambda^4)^3,$$

where $$\lambda^2\equiv 4m^2.$$

Similarly, $$z^4+3m^3z\equiv m^3z\,(z^3+4m^3)^2,$$

and $$z^5+2m^3z^2\equiv 5z^5\,(z^3-4m^3)^2.$$

M. Hermite's last reduct can be brought to a simpler form. For

$$z^5+3m^2z^3+m^3z^2+6m^4z\equiv 6m^4\left\{(z+2m)^5+4m^5\right\}^5+m^5,$$

which, without loss of generalization, can be replaced by the reduct

$$(z^5+m^5)^5.$$

I shall therefore substitute the following seven reducts in place of those given by M. Hermite:—

$$x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$x^5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$x\,(x^2+m^2)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$m^2x^5\,(x^4+m^4)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$x\,(x^3+4m^3)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$x^5\,(x^3+4m^3)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$(x^5+m^5)^5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If these be generalized in the manner indicated by M. Hermite, we shall obtain the same 5040 substitution-functions as can be derived from his reducts.

It is interesting to notice that all the reducts except (7) can be written in the form

$$x^r\,(fx^s)^{(p-1)/s},$$

where r is prime to and less than $p-1$ (*i.e.*, 1 or 5), and fx^s is a rational integral function of x^s which can never become zero, and is not a perfect power of any function of x^s, and where s is a factor of $p-1$.

It is, moreover, easy to see that such an expression always satisfies the required conditions for a substitution-quantic. For, if the reduct be raised to a power other than the s^{th}, we shall have a set of terms whose indices are of the form $ns+r$; and, since r is prime to s and $p-1$, this can never be equal to a multiple of $p-1$. Again, if it be raised to the s^{th} power, we get, by Fermat's theorem, x^{rs}, which cannot $\equiv x^{p-1}$, since r is prime to $p-1$.

The seventh reduct is obviously a substitution-function, for it is the result of the operation of θ on $\theta z+m^5$, where

$$\theta z \equiv z^5.$$

3. *Isomorphism and Vertex-shifting.*

It is proved in all works on Substitutions that ϕx and any function of the form $f^{-1}\phi fx$ represent similar or isomorphous substitutions, if fx also represents a substitution. For instance, if ϕx represent Fig. 1, § 1, then $f^{-1}\phi fx$ will represent some heptagram consisting of a pentagon and a line, and moreover $f(x)$ can be chosen to make it represent any such heptagram.

The geometrical meaning of $f^{-1}\phi fx$ can be very simply demonstrated.

Let ϕx be represented by Fig. 1. Then the congruence

$$y \equiv \phi x,$$

shows to what point any vertex x must be drawn.

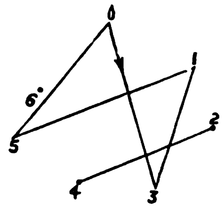

Fig. 1.

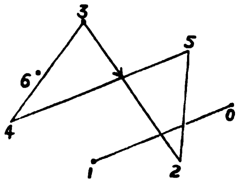

Fig. 2.

Now, suppose $f0$, $f1$, $f2$ ... have the values 3, 5, 0, 2, 1, 4, 6, and let us re-number the vertices according to this substitution, so that any vertex formerly marked a is now called fa. Now, draw the polygram of Fig. 1 exactly as before, as in Fig. 2. The method of tracing it is now given by the congruence

$$fy \equiv \phi fx,$$

for the same relation now holds between fy and fx as before held between y and x.

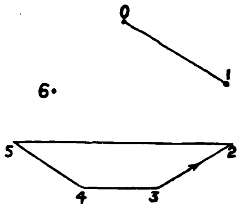

Fig. 3.

Now rearrange the vertices in their proper order, and we get Fig. 3, which is represented by

$$fy \equiv \phi fx,$$

i.e., $$y \equiv f^{-1}\phi fx.$$

The geometric method of deriving $f^{-1}\phi fx$ from ϕx shows that the two figures are isomorphous or like-membered.

With this section, *cf.* Serret, § 413.

If $$fx \equiv x+n,$$

we have to shift each vertex back n places without moving the figure, which, on restoring 0 to its proper place, means that we shift on the figure n places.

Thus $\phi(x+n)-n$ denotes the same figure as ϕx, rotated back n places.

4. *Reciprocal Polygrams.*

The most important investigation in the theory of polygrams is that of finding the reciprocal of any quantic, by which is meant that quantic whose figure is the same as that of the given quantic, with the arrow-head turned in the opposite direction. Or, analytically speaking, if

$$y \equiv \phi x,$$

where ϕ is known in form, the problem is to find the form of $\phi^{-1}x$.

Let $$y \equiv Ax^5+Bx^4+Cx^3+Dx^2+Ex,$$

so that $$\phi 0 \equiv 0,$$

and suppose $$x \equiv \alpha y^5+\beta y^4+\gamma y^3+\delta y^2+\eta y.$$

Then $$xy \equiv A+Bx^5+\dots$$
$$\equiv \alpha+\beta y^5+\dots.$$

Now, by Hermite's conditions for a valid quantic, we know that, if $x\equiv 0$, then y, y^2, y^3, y^4, y^5 all $\equiv 0$, so that we get

$$\alpha \equiv A \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Again, $$xy^2 \equiv \alpha y+\beta+\gamma y^5+\dots$$
$$\equiv x\,(Ax^5+Bx^4+\dots)^2.$$

Putting $x\equiv 0$ after reducing the last expression by Fermat, we see that

$$\left.\begin{array}{ll} \beta \equiv & \text{coefficient of } x^5 \text{ in } (\phi x)^2; \\ \gamma \equiv & \quad\text{,,} \qquad \text{,,} \qquad \text{,,} \quad (\phi x)^3 \\ \delta \equiv & \quad\text{,,} \qquad \text{,,} \qquad \text{,,} \quad (\phi x)^4 \\ \eta \equiv & \quad\text{,,} \qquad \text{,,} \qquad \text{,,} \quad (\phi x)^5 \end{array}\right\} \dots\dots\dots\dots\dots (2).$$

similarly,

Thus, if $$\phi 0 \equiv 0,$$

the reciprocal of ϕx may be directly calculated, though in the general case the process is laborious.

If $y \equiv \phi x + b$,

it is obvious that $x \equiv \phi^{-1}(y-b)$ ....(3),

so that the reciprocal of every substitution-quantic may be determined. This, by rotation through b points, can be brought into the form $\phi^{-1}y - b$.

I shall now proceed to the calculation of the reciprocals of the several standard forms given in § 2, but it will be advisable for the sake of greater generality to multiply each of these reducts by a constant, say a.

In the first place, the forms (1), (2), (7) present no difficulty,

since, if $y \equiv ax$,

then $x \equiv a^5 y$;

if $y \equiv ax^5$,

then $x \equiv ay^5$;

and if $y \equiv a(x^5+m^5)^5$,

then $y^5 \equiv a^5(x^5+m^5)$,

so that $x \equiv a^5(y^5-a^5m^5)^5$.

The reciprocal forms corresponding to the other reducts have, however, to be obtained according to the method indicated above.

Thus, if $y \equiv ax(x^2+m^2)^3$,

then $y^2 \equiv a^2x^2$,

so that $\beta \equiv \delta \equiv 0$;

$y^3 \equiv a^3x^3(x^2+m^2)^3$,

so that $\gamma \equiv a^3 . 3m^2$;

$y^5 \equiv a^5x^5(x^2+m^2)^3$,

so that $\eta \equiv a^5(1+m^6)$;

while the coefficient of x^6 in xy is $a . 3m^4$.

These results give $x \equiv a^5 y (y^2 + a^2m^2)^3$,

as may be easily verified.

Similarly, if $y \equiv am^2x^5\,(x^4+m^4)^3$,

then $x \equiv am^2y^5\,(y^4+a^4m^4)^3$.

In form (5), we have $y \equiv ax\,(x^3+4m^3)^2$,

so that $y^2 \equiv a^2x^2\,(3+x^3m^3)^2 \equiv a^2x^2\,(x^3-4m^3)^2$,

and $y^3 \equiv a^3x^3$;

and finally, applying the method explained above,

$$x \equiv a^5y\,(y^3-4a^3m^3)^2.$$

Similarly, from form (6), where

$$y \equiv ax^5\,(x^3+4m^3)^2,$$

we get $x \equiv ay^5\,(y^3+4a^3m^3)^2$,

as may be obtained from the last by changing x into x^5.

Collecting the above results, we get the following seven pairs of reciprocal equations:—

$y \equiv ax$,	$x \equiv a^5y$,	$\mu^2 \equiv a^4$,
$y \equiv ax^5$,	$x \equiv ay^5$,	$\mu^6 \equiv 1$,
$y \equiv ax\,(x^2+m^2)^3$,	$x \equiv a^5y\,(y^2+a^2m^2)^3$,	$\mu^2 \equiv a^4$,
$y \equiv am^2x^5\,(x^4+m^4)^3$,	$x \equiv am^2y^5\,(y^4+a^4m^4)^3$,	$\mu^2 \equiv a^2$,
$y \equiv ax\,(x^3+4m^3)^2$,	$x \equiv a^5y\,(y^3-4a^3m^3)^2$,	$\mu \equiv -a^5$,
$y \equiv ax^5\,(x^3+4m^3)^2$,	$x \equiv ay^5\,(y^3+4a^3m^3)^2$,	$\mu^3 \equiv a^3$,
$y \equiv a\,(x^5+m^5)^2$,	$x \equiv a^5\,(y^5-a^5m^5)^5$,	$\mu \equiv -a^5$.

One remarkable fact is to be noticed concerning these reciprocal quantics. If

$$y \equiv \phi x,$$

where ϕx is a reduct, then in every case $\phi^{-1}x$ can be reduced to the form $\mu\phi\mu x$, where μ is a constant. In other words, if

$$y \equiv \phi x,$$

then $x \equiv \mu\phi\mu y$;

that is, if $z \equiv \phi\mu y$,

then $$x \equiv \mu z,$$

so that $$y \equiv \phi\mu z;$$

which shows that $\phi\mu x$ is a self-reciprocal quantic, a species which will be treated of further on. The congruences giving the values of μ for each form are to be found in the third column above, each placed on the same line as the form to which it refers.

More generally, if $y \equiv \psi x \equiv \phi(x-a)+b,$

where ϕx is a reduct, then evidently

$$\psi^{-1}x \equiv \phi^{-1}(x-b)+a.$$

Now $$\phi^{-1}x \equiv \mu\phi\mu x;$$

therefore $$\phi^{-1}(x-b) \equiv \mu\phi\mu(x-b).$$

But $$\phi x+b \equiv \psi(x+a), \text{ by hypothesis};$$

therefore $$\phi(\mu x-\mu b)+b \equiv \psi(\mu x-\mu b+a).$$

Hence $$\psi^{-1}x \equiv \mu\psi(\mu x-\mu b+a)-\mu b+a,$$

a congruence giving a general connexion between a quantic and its reciprocal.

If $$\mu b-a \equiv \mu c,$$

then $$\psi^{-1}x \equiv \mu\psi\mu(x-c)-\mu c.$$

Let $$\psi\mu(x-c) \equiv z,$$

so that the above congruence gives us

$$\psi^{-1}x \equiv \mu(z-c).$$

Then $$x \equiv \psi\mu(z-c),$$

so that x, z are connected by a self-reciprocal relation.

Hence, if ψx represent any heptagram, two constants μ, c can always be found, such that $\psi\mu(x-c)$ is a self-reciprocal quantic.

5. *Skew-symmetry and Self-reflexion.*

A polygon is called skew-symmetric when its geometric form is its own reflexion in some line, but the direction-arrows are reversed. If the number of vertices is odd, the axis of symmetry must pass through one of the vertices. The polygon is then said to be symmetric about that vertex.

A polygon is called self-reflective when both its geometric form and the direction-arrows are reflected in some line.

The condition for self-reflexion about the zero-vertex is obviously that

$$\phi x \equiv -\phi(-x) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

that is, that ϕx should only contain odd powers of x, as is the case in the first four reducts found in § 2.

The test for skew-symmetry depends upon the form of the reciprocal quantic, the general law being that the reciprocal figure should be the exact reflexion of the original figure in some line through a vertex. If the axis pass through the zero-vertex, we get

$$\phi^{-1}x \equiv -\phi(-x) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The reciprocating constant, therefore (§ 4), is congruent with -1. We can, moreover, deduce the very important fact that, if the reduct ϕx gives a skew-symmetric figure, so also will $\phi x+b$. For the reciprocal of the latter is $\phi^{-1}(x-b)$, which, rotated back through b vertices, becomes $\phi^{-1}x-b$. But this is $-\phi(-x)-b$, which is the reflexion of $\phi x+b$, the original figure.

Now the vertex of symmetry for the reduct ϕx is zero, while, for $\phi x+b$, we must shift back the reciprocal b places before we get the reflexion of the original. A little consideration will show that the vertex of symmetry is that whose number is congruent with $\tfrac{1}{2}b$, *i.e.*, $4b$.

The skew-symmetric reducts are the following :—

$$\pm x, \quad ax^5, \quad \pm x(x^2+m^2)^3, \quad \pm m^2x^5(x^4+m^4)^3,$$

$$+x(x^3+4m^3)^2, \quad ax^5(x^3+4m^3)^2 \quad \text{where} \quad a^3 \equiv -1,$$

and $$(x^5+m^5)^5.$$

Except for $-x$, we get seven different figures from each of these forms by adding a constant b, so that the totality of skew-symmetric heptagrams* is

$$1+7+42+42+42+14+42+42 = 232.$$

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* It is interesting to note that all pentagrams can be represented by the two congruences

$$y \equiv ax+b, \quad y \equiv ax^3+b, \quad \text{mod. } 5$$

6. *Self-reciprocal or Line-and-Dot Polygrams.*

If a polygon is its own reciprocal, it is evidently made up of lines and dots.

Now, if
$$\phi^{-1}x \equiv \phi x,$$
the reciprocating constant μ is $\equiv 1$, and the self-reciprocal forms are as follows :—

$$\pm x, \quad ax^5, \quad \pm x\,(x^2+m^2)^2, \quad \pm m^2x\,(x^4+m^4)^2, \quad -x\,(x^3+4m^3)^2,$$
$$ax^5\,(x^3+4m^3)^2 \text{ where } a^3 \equiv 1, \text{ and } -(x^5+m^5)^5.$$

Their totality is
$$2+6+6+6+2+6+6 = 34.$$

It may be noticed, as is geometrically obvious, that, if ϕx be skew-symmetrical about the zero-axis, then $-\phi x$ is self-reciprocal. Moreover, if ϕx be both self-reflective and skew-symmetric, it must be also self-reciprocal.

7. *Metrical Properties. Sets.*

The seven figures corresponding to the quantic $\phi x+b$, where ϕx is a reduct, and b has the seven values 0, 1, 2 . . . , I shall call a *set* of figures, or the figures belonging to the same set. It will be found in general that figures of the same set apparently differ very much in their geometric properties. There are, however, properties possessed by each member of a set, which it is my object now to point out, and which will give us a method of detecting, by mere inspection of the geometric figure, the reduct to which the figure belongs. It is, of course, always possible to find the quantic of the given figure, and reduce it by so rotating that its second term vanishes; but this method is tedious, and furnishes no clue as to the connexion between geometric and analytic similarity.

(see SERRET's *Algebra*, § 485). Moreover, if a be not $\equiv 1$,
$$y \equiv ax+b$$
may be rotated into
$$y \equiv ax,$$
which is self-reflective ; also
$$y \equiv x+b$$
is regular, while the reciprocal of
$$y \equiv ax^3+b \quad \text{is} \quad x \equiv a\,(y-b)^3,$$
which can be rotated into
$$x \equiv ay^3-b,$$
which shows that all figures belonging to this congruence are skew-symmetrical. Combining these results, we see that—*All pentagrams formed by joining the angular points of a regular pentagon are symmetrical.*

The most important property possessed by all members (or none) of a set is skew-symmetry, as we saw in § 5.

However, since each kind of reduct was found to include skew-symmetric forms, this property will not serve to distinguish different kinds, though it will differentiate sub-species, such as are marked by the coefficient of the leading term being a residue or a non-residue.

We may, in fact, subdivide the species already found into twenty-four in all, and we may geometrically distinguish these (1) by observing the number of equal sides, and (2) by the number of parallel sides in any figure. Now, the analytical condition for equality of two or more sides is that $\phi x - x$ should have the same value for two or more values of x, while the condition for parallelism is that $\phi x + x$ should have the same value for two or more values of x. The first fact is easy to see, and the second is obvious when we consider that $\frac{1}{2}(\phi x + x)$ is the number of the vertex halfway between x and ϕx; for these two sides have the same midway vertex, that is, are parallel.

We have to notice one or two special cases :—

(1) Isolated dots must be looked upon as sides of equal length.

(2) An isolated line must be treated as a pair of parallel lines.

(3) An isolated dot midway between the extremities of a side must be considered as a side parallel to that side.

It is very easy to see that every member of a set has the same equality of sides and parallelism of sides. For, if $\phi x_1 \pm x_1$ and $\phi x_2 \pm x_2$ have equal values, so also have $\phi x_1 + c \pm x_1$ and $\phi x_2 + c \pm x_2$. This may be expressed by saying that the isoscelism or parallelism of a set is $q, r, s \ldots$ if we wish to state that the figures have each a group of q equal or parallel sides, and another group of r equal or parallel sides, not equal or parallel to the last, &c.

It may be worth noticing that there is a kind of reciprocal relation connecting equality of sides and parallelism in the figures represented by ϕx and $-\phi x$, or, as we may say, in any figure and its *negative*. For groups of parallel sides depend upon the groups of congruent values in $\phi x + x$, that is, upon groups of congruent values in $-\phi x - x$, which shows that they are equivalent to the groups of equal sides in $-\phi x$. For instance, $x^5 + 2x^3 + 5x + b$ has five equal sides for all values of b, and, consequently, $6(x^5 + 2x^3 + 5x + b)$ has five sides parallel.

8. *Descriptive Properties of sets of Polygrams.*

Besides symmetry, we have hitherto only considered the metrical properties possessed by every member of a set, *i.e.*, those properties, such as isoscelism and parallelism, which depend on geometric lengths and directions. We have now to consider a property which refers simply to the number of lesser polygons into which a polygon may split up, and which we may call a descriptive property.

The fundamental theorem is as follows :—

If ϕx consist of m members, and fx consist of n members, then $f\phi x$ or ϕfx will consist of r members, where

$$r \equiv m+n-1, \text{ mod. } 2.$$

As this is a theorem known to those who have studied the properties of substitutions, I need only refer to Mr. Asquith's paper in the *Quarterly Journal* for October, 1889, p. 114, in which the proposition stands in the form: A cycle is always added to or subtracted from substitution by a transformation. Now, every substitution can be made up of a certain number of transformations, which are represented by figures consisting of one line and $p-2$ dots, where p is the number of sides (prime) of the polygram. It therefore consists of an even number of sub-figures or members, so that we shall call it an even polygram.

Now, it is easy to infer that an odd polygram will be made up of an even number of transpositions, and *vice versâ.*

Hence the theorem, as re-worded above, follows easily.

For instance, if

$$fx \equiv x+\lambda,$$

then fx is a complete or one-membered polygram, so that ϕx and $f\phi x$—*i.e.*, $\phi x+\lambda$—are both even or both odd, or, as we may say, of *like character*. Hence all members of a set are of like character.

This may be readily verified, *e.g.*, in the case of, say, $3x^5$, which has four members, and is therefore even. It will be found that $3x^5+1$ has two, $3x^5+2$ has four, $3x^5+3$ has six, &c.

We may further notice that ϕx and $-\phi x$ have unlike characters, if

$$p = 7,$$

since, in this case, $f(x) \equiv -x,$

which consists of a dot and three lines, and is therefore even.

It is in many cases easy from analytical considerations to discover the form and character of the reducts given in § 2.

Let us first draw heptagrams corresponding to $x+\lambda$, $-x+\lambda$, $ax+\lambda$ (where a is a primitive root), and $-ax+\lambda$.

It will be found that $x+\lambda$ gives a complete heptagon; $-x+\lambda$ gives three lines and a dot; $ax+\lambda$ gives a hexagon and a dot; $-ax+\lambda$ gives two triangles and a dot.

Now, let us consider the seventh form

$$a\,(x^5+m^5)^5,$$

or, as we may write it,

$$\{a^5\,(x^5+m^5)\}^5.$$

If

$$fx \equiv x^5,$$

we may write this

$$f^{-1}\,(a^5fx+a^5m^5),$$

which, by § 3, is isomorphous with

$$a^5\,(x+m^5).$$

It is easy to see then that

$$a\,(x^5+m^5)^5$$

will represent a complete heptagon, three lines and a dot, a dot and a hexagon, or two triangles and a dot, according as $a\equiv 1$, $\equiv -1$, $\equiv$ a primitive root, or $\equiv 2$ or 4.

Again, let

$$\phi x \equiv ax\,(x^2+m^2)^3;$$

then

$$\phi^2x \equiv a^2x\,(x^2+m^2)^3\,(a^2x^2+m^2)^3 \equiv x, \text{ if } a^2\equiv 1,$$

so that

$$\pm x\,(x^2+m^2)^3,$$

gives self-reciprocal heptagrams. Moreover

$$\phi^3x \equiv a^3x\,(x^2+m^2)^3\,(a^2x^2+m^2)^3\,(a^4x^2+m^2)^3$$
$$\equiv -x, \text{ if } a^3\equiv -1.$$

Now, ϕ^3x consists of three lines and a dot, so that it is easy to see that ϕx must either consist of a dot and a hexagon, or must be self-reciprocal. The latter cannot be true unless $a^2\equiv 1$; therefore, if a be a primitive root, ϕx must consist of a dot and a hexagon, and is therefore even in character.

In similar ways we may establish the fact that if $a^3\equiv -1$ in any reduct, then the character of the corresponding heptagram is even; but it is scarcely necessary to prove the statement in the case of every reduct. It is interesting to notice that if

$$\phi x \equiv am^2x^5\,(x^4+m^4)^3,$$

then ϕ^4x in all cases $\equiv x$, so that ϕx must, if not self-reciprocal, consist partly of a quadrilateral figure.

The following Table represents twenty-four subdivisions in the different species of heptagrams, arranged according to their analytical forms, and with the geometric properties appended which are common to every member of the corresponding sets. The descriptive properties of symmetry and character have been, or can be, found directly by analysis. The metrical properties of isoscelism and parallelism have been found by inspection.

It has been found convenient to use the symbol $\sqrt[3]{1}$ for a quadratic residue, and $-\sqrt[3]{1}$ for a non-residue; a for a primitive root.

		METRICAL PROPERTIES.		DESCRIPTIVE PROPERTIES.	
		Isoscelism.	Parallelism.	Character.	Symmetry.
Form 1	x	7	0	odd	symmetrical
	$-x$	0	7	even	sym.
	$-ax$	0	0	odd	(self-reflective)
	ax	0	0	even	(self-reflective)
Form 2	$\sqrt[3]{1} . x^5$	3 . 2 . 2	2 . 2	odd	sym.
	$-\sqrt[3]{1} . x^5$	2 . 2	3 . 2 . 2	even	sym.
Form 3	$x (x^2 + m^2)^3$	3	5	odd	sym.
	$-x (x^2 + m^2)^3$	5	3	even	sym.
	$-ax (x^2 + m^2)^3$	2 . 2	2 . 2	odd	unsym.
	$ax (x^2 + m^2)^3$	2 . 2	2 . 2	even	unsym.
Form 4	$m^2x^5 (x^4 + m^4)^3$	. 2 . 2	3 . 3	odd	sym.
	$-m^2x^5 (x^4 + m^4)^3$	3 . 3	3 . 2 . 2	even	sym.
	$-am^2x^5 (x^4 + m^4)^3$	2 . 2	3	odd	unsym.
	$am^2x^5 (x^4 + m^4)^3$	3	2 . 2	even	unsym.
Form 5	$x (x^3 + 4m^3)^2$	2 . 2 . 2	0	odd	sym.
	$-x (x^3 + 4m^3)^2$	0	2 . 2 . 2	even	unsym.
	$-ax (x^3 + 4m^3)^2$	4	2 . 2 . 2	odd	unsym.
	$ax (x^3 + 4m^3)^2$	2 . 2 . 2	4	even	unsym.
Form 6	$\sqrt[3]{1} . x^5 (x^3 + 4m^3)^2$	3	3 . 2	odd	unsym.
	$-\sqrt[3]{1} . x^5 (x^3 + 4m^3)^2$	3 . 2	3	even	sym.
Form 7	$(x^5 + m^5)^5$	4 . 2	2 . 2	odd	sym.
	$-(x^5 + m^5)^5$	2 . 2	4 . 2	even	unsym.
	$-a (x^5 + m^5)^5$	2 . 2	3 . 2	odd	unsym.
	$a (x^5 + m^5)^5$	3 . 2	2 . 2	even	unsym.

December 11*th*, 1890.

Prof. GREENHILL, F.R.S., President, in the Chair.

The following gentlemen were elected members :—F. S. Carey, M.A., late Fellow of Trinity College, Cambridge, Professor of Mathematics, University College, Liverpool; M. W. J. Fry, M.A., Fellow of Trinity College, Dublin; H. S. Romer, M.A., late Scholar of Trinity Hall, Cambridge; and Hari Dâs Sastri, M.A., Director of Public Instruction, Jaypur State, Rajputana.

The Auditor made his Report. Upon the motion of Sir J. Cockle, seconded by Mr. S. Roberts, the Treasurer's Report was then adopted. A vote of thanks was unanimously accorded to Mr. Heppel for the trouble he had taken in auditing the accounts.

The following communications were made :—

On the Stability of a Plane Plate under Thrusts in its own Plane, with applications to the "Buckling" of the Sides of a Ship: Mr. G. H. Bryan (communicated by Mr. Love).

On the Extension to Matrices of any Order of the Quaternion Symbols S and V: Dr. Taber.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables: Mr. E. B. Elliott.

Newton's Classification of Cubic Curves: Mr. W. W. R. Ball.

Steiner's Poristic Systems of Spheres: Prof. G. B. Mathews.

On the q-series derived from the Elliptic and Zeta Functions of $\frac{1}{3}k$ and $\frac{1}{4}k$: Dr. Glaisher.

The following presents were received :—

"Educational Times," for December.

"Proceedings of the Physical Society of London," Vol. x., Part iv.; Nov., 1890.

"Proceedings of the Cambridge Philosophical Society," Vol. vii., Part ii.

"Nautical Almanack," for 1894.

"Bulletin des Sciences Mathématiques," Tome xiv., Nov., 1890.

"Nieuw Archief voor Wiskunde," Deel xvii., Stuk 1 and 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vi., Fasc. 4, 6, and 7; Roma, 1890.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 116, 117, and 118.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xx.-xl.

"Acta Mathematica," xiii., 1 and 2.

"Memorias de la Sociedad Científica—Antonio Alzate," Tomo III., Nos. 11 and 12.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 6.

Pamphlets by A. Voss:—"Ueber die mit einer bilinearen Form vertauschbaren bilinearen Formen," 8vo; "Ueber die conjugirte Transformation einer bilinearen Form in sich selbst," 8vo; "Ueber einen Satz aus der Theorie der Determinanten," 8vo; "Ueber die cogredienten Transformationen einer bilinearen Form in sich selbst," 4to.

On the Stability of a Plane Plate under Thrusts in its own Plane, with Applications to the "Buckling" of the Sides of a Ship.

By G. H. BRYAN.

[*Read Dec.* 11*th*, 1890.]

Introduction.

1. The problems discussed in this paper are the analogues for a plane rectangular or circular plate of the well-known investigations of the stability of a thin wire or shaft, due in the first place to Euler, and since developed by Greenhill. I have employed the energy criterion of stability, the use of which I have already illustrated in this connexion in two papers published in the *Proceedings of the Cambridge Philosophical Society*.*

The case of a plate supported on equidistant parallel ribs will be considered more fully, on account of the practical use of such structures in the construction of ships.

Suppose a plane elastic plate is submitted to edge tractions in its own plane which produce compression of its middle surface, and let every point of that surface receive a displacement normal to the plane, such displacements being chosen in accordance with the prescribed boundary conditions. If this displacement be everywhere of the first order of small quantities, the surface of the plate will thereby become extended by small quantities of the second order, and the strains produced in the surface will also be thereby diminished by quantities of the second order. Multiplying these strain variations by the corresponding stresses, and integrating over the surface, we find the work done by the stresses. This work is the loss of potential energy consequent on stretching, or, more strictly, on diminution of the compression originally produced in the middle surface by the given edge tractions. If this loss is greater than the corresponding gain of potential energy dependent on the bending of the surface, the total energy will be greater in the plane form, which will therefore be a form of unstable equilibrium. The plate will then be liable to "buckle up," and corrugations will appear on its surface.

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* *Camb. Phil. Proc.*, Vol. VI., pp. 199, 286.

If equilibrium is "critical," the displaced form is also in equilibrium, and may be investigated by the method of variation.

General Theory of the Stability of a Plane Plate.

2. Take any rectangular axes of x and y in the plane of the plate, and let the stresses due to the edge tractions at a point (x, y) of the surface consist of a thrust T_1 (per unit length) parallel to x, a thrust T_2 parallel to y, and a shearing stress of magnitude M. Let the plate be displaced in such a manner that the point (x, y) receives a displacement w perpendicular to the plane of the plate. Let σ_1, σ_2 be the resulting extensions parallel to the axes of x, y at the point, ϖ the shear of the angle between them. If ds is an element of length, measured between any contiguous points on the deformed surface, then evidently

$$ds^2 = dx^2 + dy^2 + dw^2.$$

But, since w is a function of x and y,

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy,$$

whence, substituting,

$$ds^2 = \left\{1 + \left(\frac{\partial w}{\partial x}\right)^2\right\} dx^2 + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} dx\,dy + \left\{1 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dy^2.$$

Also, $$ds^2 = (1+\sigma_1)^2 dx^2 + 2\varpi\, dx\,dy + (1+\sigma_2)^2 dy^2;$$

therefore, neglecting powers of w after the second,

$$2\sigma_1 = \left(\frac{\partial w}{\partial x}\right)^2, \quad 2\varpi = 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \quad 2\sigma_2 = \left(\frac{\partial w}{\partial y}\right)^2.$$

The *loss* of potential energy due to extension, or the work done by the stresses, is equal to W, where

$$W = \iint (T_1\sigma_1 + M\varpi + T_2\sigma_2)\,dx\,dy$$

$$= \tfrac{1}{2}\iint \left\{ T_1 \left(\frac{\partial w}{\partial x}\right) + 2M\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} + T_2\left(\frac{\partial w}{\partial y}\right)^2 \right\} dx\,dy \quad \text{...(1).}$$

Let $2h$ be the thickness of the plate, E Young's modulus, σ Poisson's ratio, and let β be given by

$$\beta = \tfrac{2}{3}h^3\frac{E}{1-\sigma^2} \quad \text{..........................(2).}$$

Then, if V be the potential energy due to bending, we have

$$V = \tfrac{1}{2}\beta\iint \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right] dx\,dy$$

$$\text{.................(3).}$$

The condition of stability is that V must be greater than W for every possible type of displacement. If for any displacement $V < W$, the plane form will be unstable, and the plate will buckle up. The number of corrugations produced when this happens will be determined by the form of w as a function of x and y.

Supported Rectangular Plate.

3. Consider the case of a rectangular plate bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$, and supported at these edges so as to prevent lateral motion. Suppose also that the given edge tractions are uniform and perpendicular to the edges, being T_1 per unit length on the edges $x = 0$, $x = a$, acting parallel to the axis of x, and T_2 on the edges $y = 0$, $y = b$, parallel to the axis of y. Then, in the expression for W, $M = 0$, and T_1, T_2 are constant over the plate, having the values given at the edges.

Since the edges are supported so that the normal displacement w is zero round the boundary, it follows that w can be expanded by Fourier's theorem in the form

$$w = \sum\sum A_{m,n}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad \text{.....................(4).}$$

Therefore

$$W = \tfrac{1}{2}T_1\int_0^b\int_0^a \left\{\sum\sum A_{mn}\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\right\}^2 dx\,dy$$

$$+\tfrac{1}{2}T_2\int_0^b\int_0^a \left\{\sum\sum A_{mn}\frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\right\}^2 dx\,dy$$

$$= \tfrac{1}{8}ab\pi^2\sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2 \qquad \text{.........................(5).}$$

In like manner, we find on integration

$$V = \tfrac{1}{8}ab\pi^4\beta\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2A_{mn}^2 \qquad \text{................ (6).}$$

The plane form will therefore be stable, provided that, for all values of the constants A_{mn},

$$\pi^2\beta\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2A_{mn}^2 > \sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2,$$

or

$$\pi^2\beta > \frac{\sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2}{\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2A_{mn}^2} \qquad \text{................(7).}$$

Suppose that the thrusts T_1, T_2 are given, then inequality (7) determines the limit to the value of β, and hence, the limiting thickness of the plate in order that it may just not buckle. To find this limit, we must choose the constants A_{mn} so that the right-hand side of (7) is a maximum. This will be the case when all but one of the constants are zero, and, if A_{mn} be this constant, the limiting value of β will be given by

$$\pi^2\beta = \frac{T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \qquad \text{........................(8),}$$

the numbers m, n being so chosen as to make this expression for β the greatest possible.

The corresponding form of the displaced surface is given by

$$w = A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{......................(9).}$$

w vanishes along the lines

$$x = a/m, \quad x = 2a/m, \quad x = 3a/m, \ \&c.,$$

and also along the lines

$$y = b/n, \quad y = 2b/n, \quad y = 3b/n, \ \&c.$$

Hence it readily follows that, if the thickness of the plate be less than that given by (8), the surface will buckle up, and the corrugations will divide it into mn rectangles, over which the displacement is alternately to one side and to the other.

We shall see hereafter that one of the numbers m, n must be unity, so that the corrugations can only divide the plate into a single row of rectangles. The methods of determining the number of corrugations are, however, better shown by first considering the following special cases.

4. Let $T_1 = T_2$. Put each of these $= T$.

Equation (8) becomes

$$\pi^2\beta = \frac{T}{\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}} \quad \text{..........................(10).}$$

β is greatest when m and n are least, that is, when $m = 1$, $n = 1$. Therefore the limit for β is given by

$$\pi^2\beta = \frac{T}{\dfrac{1}{a^2} + \dfrac{1}{b^2}}.$$

If the plate be too thin, its initial form on buckling will be found by putting $m = 1$, $n = 1$ in (8). We see that there will be a single corrugation, all points being displaced towards the same side.

5. Let $T_2 = 0$.

Equation (8) becomes

$$\pi^2\beta = \frac{T_1 \dfrac{m^2}{a^2}}{\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \quad \text{.......................(11).}$$

The right-hand side is greatest when $n = 1$, but the same is not *necessarily* true as regards m. The number (m) of corrugations parallel to the side a will depend on the length a. The greatest length a of a plate which will buckle into m corrugations will be that for which the limiting values of β, corresponding to m and $m+1$ corrugations, respectively, are equal. Therefore, putting $n = 1$ in (11), we must have for this greatest length

$$\frac{\dfrac{m^2}{a^2}}{\left(\dfrac{m^2}{a^2}+\dfrac{1}{b^2}\right)^2} = \frac{\dfrac{(m+1)^2}{a^2}}{\left(\dfrac{(m+1)^2}{a^2}+\dfrac{1}{b^2}\right)^2}.$$

Put $a/b = k$, so that $a = kb$. The last equation gives

$$\frac{m^2}{\{m^2+k^2\}^2} = \frac{(m+1)^2}{\{(m+1)^2+k^2\}^2};$$

whence $\quad m\{(m+1)^2+k^2\} = \pm(m+1)\{m^2+k^2\}.$

Since k^2 is positive, we must take the upper sign to the radical.

This gives $\quad k^2 = m(m+1),$

whence $\quad a = b\sqrt{\{m(m+1)\}} \quad\text{.....................(12)}.$

Similarly, the least length a for which the plate, if unstable, will buckle into m corrugations parallel to the side a, will be given by

$$a = b\sqrt{\{m(m-1)\}} \quad\text{.....................(13)}.$$

Hence there will be m corrugations parallel to the side a, provided that

a lies between the values $b\sqrt{\{m(m-1)\}}$ and $b\sqrt{\{m(m+1)\}}$.

Condition (12) may also be written

$$\frac{a}{m} : b = b : \frac{a}{m+1},$$

which shows that, whether the corrugations divide the plate into m or $m+1$ rectangles, the ratio of their length to their breadth will be the same in both cases, but the directions of the longer and shorter sides of the rectangles will be interchanged.

Hence it readily follows that the number of corrugations must always be such that the ratio of the longer to the shorter side of the rectangles differs as little as possible from unity, that is, their shape must be as nearly square as possible.

Hence it is evident that if the length of the plate become infinite, the corrugations will divide it exactly into squares.

This may also be shown as follows :—

Supposing the strip infinitely long, or $a = \infty$, let $\lambda = a/m$, so that λ is the length of a corrugation. Then (11) gives

$$\pi^2\beta = \frac{\dfrac{T_1}{\lambda^2}}{\left(\dfrac{1}{\lambda^2}+\dfrac{1}{b^2}\right)^2},$$

in which the right-hand side must be made a maximum by the variation of λ. By the ordinary methods we find that this is the case when $\lambda = b$.

6. As the collapse is of a different nature in the two cases above considered, we will now investigate the limits to the ratio of T_2 to T_1, in order that any form of the plate may buckle up in a series of corrugations. Let the length a be infinite, and let λ be the length of a corrugation. Putting $n = 1$, $a/m = \lambda$ in (8), the critical value of β is found from the relation

$$\pi^2\beta = \frac{T_1\dfrac{1}{\lambda^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{1}{\lambda^2}+\dfrac{1}{b^2}\right)^2} \quad \text{......................(14)},$$

by making β a maximum by the variation of λ.

Write $b/\lambda = \mu$, $T_2/T_1 = r$; then we must make the expression

$$\frac{\mu^2+r}{(\mu^2+1)^2}$$

a maximum by the variation of μ. Differentiating with respect to μ^2, the condition is

$$\frac{1}{(\mu^2+1)^2}-\frac{2(\mu^2+r)}{(\mu^2+1)^3}=0,$$

whence $$\mu^2+1 = 2(\mu^2+r),$$

or $$\mu^2 = 1-2r,$$

that is, $$b^2/\lambda^2 = 1-2T_2/T_1 \quad \text{......................(15)}.$$

In order that this may give a real value of λ, we must have

$$T_2 < \tfrac{1}{2}T_1 \quad\text{.................................(16)},$$

and if this is not the case, the value of β will be greatest when $\lambda = \infty$. If the inequality (16) is satisfied, equation (15) determines the length of the corrugations produced in the case of collapse. We notice that λ diminishes as the ratio of T_2 to T_1 diminishes. If T_2 becomes negative, so that the strip is acted on by lateral *tension* combined with the longitudinal thrust, the length λ of the wrinkles is less than b, and it diminishes as the lateral tension increases.

The last-mentioned property may be easily illustrated by wetting a sheet of paper in the middle, and then stretching it over two parallel rulers. The moisture causes the surface of the paper to expand and wrinkle, and if the rulers be pulled apart with increasing force, the wrinkles will become finer and closer.

We also see (as previously stated) that a rectangular plate cannot buckle up into a *network* of rectangles—*i.e.*, that m, n cannot both be different from unity. For this would require $T_2 < \frac{1}{2}T_1$ and $T_1 < \frac{1}{2}T_2$, which are incompatible.

7. If $T_2 < \frac{1}{2}T_1$, and we substitute for λ from (15) in (14), we find

$$\pi^2\beta = \frac{b^2}{4}\,\frac{T_1^2}{T_1 - T_2} \quad\text{...........................(17)},$$

which determines the least thickness of an infinite strip of breadth b, supported at its edges in order that it may resist an end-thrust T_1, and a lateral thrust T_2.

In particular, if $T_2 = 0$, we must have

$$\pi^2\beta = \frac{b^2}{4}\,T_1 \quad\text{.................................(18)}.$$

If $T_2 > \frac{1}{2}T_1$, we must put $\lambda = \infty$ in (14), and we obtain

$$\pi^2\beta = b^2 T_2 \quad\text{...................................(19)},$$

which is independent of T_1, as it evidently should be, for the corresponding displaced surface is given by

$$w = A \sin\frac{\pi y}{b}.$$

Here w is independent of x; hence no work is done by the thrust T_1 during the displacement.

If the length of the strip be finite, the case will be different. The limit for β will always depend on the values both of T_1 and T_2.

We notice that, when $T_2 = \frac{1}{2}T_1$, the values of β given by (17) and (19) are equal, as they should be.

Comparing (19) with (18), we see that the strip is capable of resisting four times as much simple longitudinal thrust as of lateral thrust.

We also see that, even when the lateral thrust is the greatest which the strip can resist, it is still possible to subject it, in addition, to any amount of longitudinal thrust on its ends, not greater than double the lateral thrust, without diminishing the strength.

Applications to the Sides of a Ship.

8. The preceding results are applicable, without any modification whatever, to the case of an infinite plate supported on parallel ribs at distances b apart. This kind of structure is met with in the sides of ships, a fact which adds considerably to the practical interest of problems of this class.

We see that, in order to obtain the greatest strength, the ribs must be placed parallel with the direction of the greatest thrust, and that, to obtain the same strength, they may then be placed twice as far apart as if they were perpendicular to the thrust.

The strength of the plate will not then be increased or altered in any way by the addition also of a second set of ribs, perpendicular to the first, and at distances apart equal to the natural length of the corrugations. Thus, if there is no lateral thrust, a plate supported on parallel longitudinal ribs will not be strengthened by the addition of transverse ribs which divide it into squares. If the transverse ribs are at any other distance apart, however, the system will be strengthened. In this particular case, the most effective distance between the transverse ribs will be $b\sqrt{2}$, because [putting $m = 1$ in (12)] each rectangle formed by the framework will buckle with equal facility into one or two corrugations, and the length of these corrugations will then differ as much as possible from their natural length.

Putting $T_2 = 0$, $m = n = 1$, $a = b\sqrt{2}$ in (8), we find

$$\pi^2\beta = \tfrac{2}{9}b^2T_1,$$

and the same increase in strength could be obtained much more economically by diminishing the distance between the ribs from b to $b\sqrt{\frac{8}{9}}$, *i.e.*, $\frac{2}{3}\sqrt{2}\,.\,b$.

When, however, the structure is subject to thrusts of about equal intensity in all directions, the advantage of a network of ribs will be much greater. Moreover, the framework itself will be strengthened —a consideration not brought out in our present theoretical treatment, in which the framework is supposed perfectly rigid.

9. The *general* problem, to determine the number of corrugations produced in the buckling of a finite rectangular plate (sides a, b) under the influence of both thrusts (T_1, T_2, where $T_1 > 2T_2$), may now be considered. In the critical case when the plate will buckle with equal facility into m or $m+1$ rectangles in the direction of its length, we have

$$\frac{T_1\dfrac{m^2}{a^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{m^2}{a^2}+\dfrac{1}{b^2}\right)^2}=\frac{T_1\dfrac{(m+1)^2}{a^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{(m+1)^2}{a^2}+\dfrac{1}{b^2}\right)^2}.$$

Whence, on reduction, we find

$$\frac{T_1}{a^4}m^2(m+1)^2+\frac{T_2}{a^2b^2}\{2m(m+1)+1\}-\frac{T_1-2T_2}{b^4}=0 \text{......(20)}.$$

Solving as a quadratic in a^2/b^2, we find

$$\frac{a^2}{b^2}=\frac{\{2m(m+1)+1\}T_2\pm\sqrt{\{4m^2(m+1)^2(T_1-T_2)^2+(2m+1)^2T_2^2\}}}{T_1-2T_2}$$

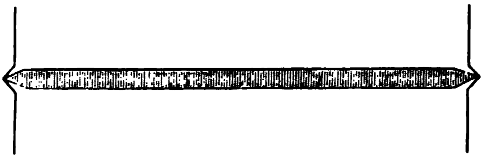
..................(21).

In accordance with previous results, this will give a real value for the ratio of a to b, provided that $T_1 > 2T_2$, and that the radical is taken with the positive sign.

10. A few words on the mode of supporting the boundaries may not be out of place. In his *Theory of Sound*, Vol. I., § 225, Lord Rayleigh gives the following suggestion as a means of realizing this kind of support:—"We may consider the plate to be held in its place by friction against the walls of a cylinder circumscribed closely round it."

This method is strictly applicable to the problems of the present paper, where the normal reaction of the cylinder is supposed to produce thrusts in the plane of the plate. But it is *not* applicable to the problem considered by Lord Rayleigh, viz., the determination of the frequencies of vibration. Obviously, the effect of the thrusts will be generally to lower the pitch of all the vibrations, and if the thrusts be increased till the plane form is in critical equilibrium, one of the frequencies will then vanish.

The annexed diagram shows a better means of support. The boundaries, being brought to sharp edges, are made to fit in grooves of a rather wider angle.



The sides of a ship are rivetted to the ribs, and this implies more constraint than that afforded by mere support. This constraint more closely resembles "clamping."

Application of Variational Method.—Clamped Circular Plate.

11. In applying the energy method to the rectangular plate, we supposed w expanded by Fourier's series, and the success of our method depended on the fact that the expression for the total energy contained no products of the coefficients. In other problems it may not be easy to discover the form of the functions in which we must expand; we therefore use another method, more analogous to that adopted by Greenhill in his paper on the "Stability of Shafting."[*] If equilibrium in the plane form is *critical*, there must be another form of equilibrium indefinitely near to it. To find this, we apply the method of variation of energy, and the boundary conditions then lead to a relation which must be satisfied in the critical case. This equation determines the criteria of stability.

12. The method of variation may be illustrated by its application to the critical equilibrium of a circular plate acted on by a normal

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[*] *Proc. Inst. Mech. Eng.*, 1883.

thrust T (per unit length) round the circumference. The slightly displaced form being also in equilibrium, the displacement is found by making

$$\delta V - \delta W = 0$$

for all variations of w.

We take as our example the simplest case, in which the boundary of the plate is clamped so as to allow of no variation of position or direction. Let the radius of the plate be a.

Putting

$$T_1 = T_2 = T \quad \text{and} \quad M = 0$$

in (1), we have

$$W = \tfrac{1}{2} T \iint \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} dx\,dy;$$

therefore

$$\delta W = T \iint \left\{ \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} \right\} dx\,dy$$

$$= T \int \delta w \frac{\partial w}{\partial r}\, ds - T \iint \delta w\, \nabla^2 w\, dx\,dy.$$

Also,

$$\delta V = \beta \iint \delta w\, \nabla^4 w\, dx\,dy + (\text{line integrals taken round the boundary}).$$

Since the plate is clamped, we have, round the boundary,

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22);$$

thus, the line integrals all vanish, and the differential equation for w is

$$\beta \nabla^4 w + T \nabla^2 w = 0.$$

Writing $\kappa^2 = T/\beta$, this becomes

$$\nabla^2 (\nabla^2 + \kappa^2)\, w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

If this be transferred to polar coordinates, a solution which is not infinite at the centre is

$$w = \cos n\theta \left\{ Ar^n + BJ_n(\kappa r) \right\} \ldots\ldots\ldots\ldots\ldots\ldots (24).$$

The boundary conditions (22) give, when $r = a$,

$$\left.\begin{aligned} Aa^n \;+\; BJ_n(\kappa a) &= 0 \\ nAa^{n-1} + \kappa BJ'_n(\kappa a) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Eliminating the ratio $A : B$, we have

$$\begin{aligned} 0 &= nJ_n(\kappa a) - \kappa a J'_n(\kappa a) \\ &= \kappa a J_{n+1}(\kappa a), \end{aligned}$$

by the well-known relations between Bessel's Functions. Therefore, for critical equilibrium corresponding to the displacement of the form (24), we must have

$$J_{n+1}(\kappa a) = 0 \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots(26),$$

and we must choose n so that the value of κ given by this equation is the least possible.

If the plate be free at the centre, we may take

$$n = 0,$$

and the least solution of the equation

$$J_1(\kappa a) = 0$$

is known to be

$$\kappa a = 3{\cdot}832\ldots.$$

Therefore $$T = 14{\cdot}684 \ldots \times \beta/a^2.$$

If the plate be supported at the centre, the solution (24) corresponding to

$$n = 0$$

cannot make w vanish, both when

$$r = 0,$$

and when $$r = a.$$

We must therefore take $$n = 1,$$

and we have $$J_2(\kappa a) = 0,$$

whence $$\kappa a = 5{\cdot}135 \ldots,$$

and $$T = 26{\cdot}368 \ldots \times \beta/a^2.$$

If the boundary be supported or free, the solution of the differential [illegible]uation for w is still of the form given by (24), but the boundary [illegible]nditions will lead to a far more complicated equation for κ.

A very good illustration of the buckling of a circular plate is frequently afforded by the lid of a circular canister, in which the thrust is due to the tension of the rim. The "dint" in such a lid can be readily pushed from one side to the other, but it is impossible to keep the surface flat, as that position is unstable.

The same principle is also illustrated in the "castanets," in which a "clicking" sound is produced by pushing a disc of metal from one side to the other of the unstable plane form.

13. In all the cases discussed in this paper, the stresses in the surface are proportional to β; and, therefore, to the cube of the thickness of the plate. Since these stresses are distributed over the thickness of the plate, the strains they produce are proportional to the square of the thickness. If, therefore, the plate be thin, these strains will be small, and there will be no rupture of the material accompanying the buckling. This accords with the general results obtained in my paper "On the Stability of Elastic Systems."*

In a future paper, I hope to deal with further applications of the variational method, with special reference to the stability of a rectangular plate or strip in certain cases when the shear M does not vanish, and when the boundary conditions are different to those assumed in the present communication.

On the Application to Matrices of any Order of the Quaternion Symbols S and V. *By* Henry Taber, Docent in Clark University, Worcester, Mass. U.S.A.

[*Read Dec.* 11*th*, 1890.]

1. *Properties of the Symbols S and V.*

The conception of scalar and vector parts of a quaternion, or matrix of the second order, may be extended to matrices of any order.† Regarded as a matrix, the scalar of any quaternion is one half the sum of its latent roots; following this analogy, I shall define the scalar of any matrix m of order ω as the ω^{th} part of the sum of its latent roots; and I shall denote this function of m, as in quaternions, by Sm. But then, since the sum of the latent roots of a matrix is the sum of the elements along its principal diagonal, it follows, if n is any other matrix of order ω, that

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* *Camb. Phil. Proc.*, Vol. vi., p. 204.

† See paper by author on the "Theory of Matrices," *Amer. Journ. Math.*, Vol. xii.

$$S(m+n) = Sm + Sn.$$

The vector part of m (denoted by Vm) is most simply defined as the matrix, less the ω-th part of the sum of its latent roots; and so defined V is also a distributive symbol of operation; *i.e.*,

$$\begin{aligned} V(m+n) &= (m+n) - S(m+n) = (m-Sm)+(n-Sn) \\ &= Vm + Vn. \end{aligned}$$

Following immediately from the distributive character of S and V, we have, as in quaternions,

$$dSm = Sdm, \quad dVm = Vdm.$$

Obviously, as in quaternions,

$$\begin{aligned} S\,.\,mn &= S\,(Sm+Vm)(Sn+Vn) \\ &= Sm\,.\,Sn + S\,.\,Vm\,Vn.^{*} \end{aligned}$$

Therefore

$$\begin{aligned} V.\,mn &= V\,(Sm+Vm)(Sn+Vn) \\ &= Sm\,.\,Vn + Sn\,.\,Vm + V.\,Vm\,Vn. \end{aligned}$$

If

$$m = \left(\begin{array}{lll} a_{11}, & a_{12}, & \&c. \\ a_{21}, & a_{22}, & \&c. \\ \&c. & & \end{array}\right), \qquad n = \left(\begin{array}{lll} b_{11}, & b_{12}, & \&c. \\ b_{21}, & b_{22}, & \&c. \\ \&c. & & \end{array}\right),$$

then

$$\begin{aligned} S\,.\,mn &= \Sigma_r a_{1r} b_{r1} + \Sigma_r a_{2r} b_{r2} + \&c. \\ &= \Sigma_r a_{r1} b_{1r} + \Sigma_r a_{r2} b_{2r} + \&c. \\ &= S\,.\,nm. \end{aligned}$$

Consequently, *the scalar of the product of any number of given matrices is unaltered by a cyclic interchange of these matrices.*

E.g.,

$$S\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}\right\}$$

$$= (aa'a'' + dd'd'') + (ab'c'' + a'b''c + a''bc') + (bc''d' + b'cd'' + b''c'd).$$

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[*] For $SVm = S(m-Sm) = Sm - Sm = 0$.

2. *The catena of identical equations.*

The term "catena of identical equations" was employed by Sylvester (*Johns Hopkins Univ. Circs.*, Vol. III.) to denote the $\omega+1$ identical relations (including the "identical equation" of either matrix) existing between two matrices of order ω which he obtained, with the aid of the latent function of the corpus of two matrices, by successive differentiations of the identical equation of either matrix. Thus, if m and n are two matrices of the second order, and the latent function of the corpus m, n, namely,

$$\text{Det. } (x+ym+zn) = x^2+2bxy+2cxz+dy^2+2eyz+fz^2,$$

the catena of m, n consists of the three identities

$$m^2-2bm+d = 0,$$

$$mn+nm-2bn-2cn+2e = 0,$$

$$n^2-2cn+f = 0.$$

The introduction into the general theory of matrices of the quaternion symbol S may be utilized to find the coefficients in the catena of equations without the aid of Sylvester's latent function of the corpus of the matrices which appear in the chain. In a paper to appear in No. 2, Vol. XIII., of the *Amer. Journ. Math.*, I have given, for matrices of the second and third orders, by employing the symbol S, the coefficients in the catena in terms of the scalars (or sums of the latent roots) of the products of powers of the matrices involved; and the method employed there may be extended to matrices of any order, though the extension of this method to matrices of the fifth or of higher orders is not very easy. In a somewhat different and simpler manner, I show here how to obtain the coefficients in the catena for matrices of the fourth order. Thus, if the identical equation in m is

$$m^4-p_1m^3+p_2m^2-p_3m+p_4 = 0,$$

we may replace the p's by their values in terms of the sums of the powers of the latent roots of m. But since, by the law of latency, the latent roots of m^r are the r^{th} powers of the latent roots of m, hence $4Sm^r =$ sum of the r^{th} powers of the latent roots of m.

Consequently $$p_1 = 4Sm,$$

$$1.2.p_2 = \begin{vmatrix} 4Sm, & 1 \\ 4Sm^2, & 4Sm \end{vmatrix} = 4\,(4S^2m - Sm^2),$$

$$1.2.3.p_3 = \begin{vmatrix} 4Sm, & 1, & 0 \\ 4Sm^2, & 4Sm, & 2 \\ 4Sm^3, & 4Sm^2, & 4Sm \end{vmatrix} = 8\,(8S^3m - 6Sm^2 . Sm + Sm^3),$$

$$1.2.3.4.p_4 = \begin{vmatrix} 4Sm, & 1, & 0, & 0 \\ 4Sm^2, & 4Sm, & 2, & 0 \\ 4Sm^3, & 4Sm^2, & 4Sm, & 3 \\ 4Sm^4, & 4Sm^3, & 4Sm^2, & 4Sm \end{vmatrix}$$

$$= 8\,(32S^4m - 48S^2m . Sm^2 + 6S^2m^2 + 16Sm . Sm^3 - 3Sm^4),$$

where S^2m or $S^2(m)$, means $(Sm)^2$ as $\sin^2 x$ means $(\sin x)^2$.

It follows from a property of the symbol S, namely

$$dSm = Sdm,$$

that we may immediately differentiate the identity in m to obtain the first derived identical equation (the second of the chain). Thus, if $dm = n$, we have

$$\Sigma m^3n - (dp_1 . m^3 + p_1 \Sigma m^2n) + (dp_2 m^2 + p_2 \Sigma mn)$$
$$- (dp_3 . m + p_3 m) + dp_4 = 0,$$

where $\Sigma m^r n$ denotes the sum of all possible products of n with r m's, *i.e.*, Σm^3n means $nm^3 + mnm^2 + m^2nm + m^3n$, and so in other cases; and

$$dp_1 = 4Sn,$$
$$dp_2 = 4\,(4Sm . Sn - S . mn),$$
$$dp_3 = 4\,(8S^2m . Sn - 4S . mn . Sm - 2Sm^2 . Sn + S . m^2n),$$
$$dp_4 = \tfrac{4}{3}\,(32S^3m . Sn - 24S . mn . S^2m - 24Sm^2 . Sm . Sn + 6Sm^2 . S . mn$$
$$+ 12S . m^2n . Sn + 4Sm^3 . Sn - 3S . m^3n).$$

In the next differentiation, regarding n as constant, we may either put the new $dm = n$, and thus obtain the second equation of Sylvester's catena; or we may put the new $dm = p$, and proceeding in this way ultimately obtain an identical relation between any four matrices.

This method of deriving the coefficients of the catena is, I think, somewhat simpler than to derive them from the latent function of the corpus of the matrices which appear in the catena; and has this advantage over that method, that these coefficients are expressed immediately as simple functions of the sums of the latent roots of the products of the powers of the matrices involved in the catena.

Obviously this process may be extended to matrices of any order.

3. *Application of the symbol S to a problem of Sylvester's.*

In the *Johns Hopkins Univ. Circs.*, Vol. I., p. 241, and Vol. III., p. 7, Sylvester has considered the problem to determine the conditions necessary and sufficient that for two nonions m and n,

$$m^3 = n^3 = 1, \quad nm = \lambda mn,$$

where λ is an imaginary cube root of unity. Sylvester finds the necessary and sufficient conditions to be

$$\text{Det. } (x+ym+zn) = x^3+y^3+z^3$$

(involving nine conditions), and that the coefficients of mn in the identical equation of mn shall vanish. Sylvester has also given a special solution.* These ten conditions leave $18-10 = 8$ arbitrary constants in the general solution. But if m, n are expressions linear in terms of $(1, i, i^2 \between 1, j, j^2)$ (i and j being special solutions), so that

$$m = (a_1, a_2, \ldots a_9 \between 1, i, i^2 \between 1, j, j^2),$$

$$n = (b_1, b_2, \ldots b_9 \between 1, i, i^2 \between 1, j, j^2),$$

Sylvester states that he is unable to express the conditions in terms

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* If i and j are the two special solutions, so that

$$i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix},$$

the general solution is given by $m = \varpi i \varpi^{-1}$, $n = \varpi j \varpi^{-1}$, where $|\varpi| \neq 0$ (see *American Journal of Mathematics*, Vol. XII., p. 390).

of these coefficients*. In what follows I shall show how this may be accomplished.

If $m^3 = 1$, $n^3 = 1$, hence $Sm^3 = 1$, $Sn^3 = 1$. If, in addition, $nm = \lambda mn$, hence $m = \lambda n^2 mn$, and $m^2 = \lambda n^2 mnm = \lambda^2 n^2 m^2 n$; and, consequently,

$$Sm = \lambda S \,.\, n^2 mn \;= \lambda S \,.\, n^3 m \;= \lambda Sm,$$

$$Sm^2 = \lambda^2 S \,.\, n^2 m^2 n = \lambda^2 S \,.\, n^3 m^2 = \lambda^2 Sm^2;$$

i.e., $Sm = 0$, $Sm^2 = 0$. Similarly, $Sn = 0$, $Sn^2 = 0$. Since $m = Vm$, $n = Vn$, it seems preferable to denote m and n, as in quaternions, by Greek letters, so I shall put $m = \alpha$, $n = \beta$; *i.e.*, the problem is: find α, β cubic matrices, such that $\alpha^3 = 1$, $\beta^3 = 1$, $\beta\alpha = \lambda\alpha\beta$, λ being an imaginary cube root of unity. From $\beta\alpha = \lambda\alpha\beta$, whence follows $S \,.\, \alpha\beta = S \,.\, \beta\alpha = 0$, we also get

$$\beta\alpha \,.\, \beta\alpha = \lambda^2 \alpha\beta \,.\, \alpha\beta, \quad \beta\alpha^2 = \lambda^2\alpha^2\beta, \quad \beta^2\alpha = \lambda^2\alpha\beta^2;$$

$$\therefore \quad S(\beta\alpha)^2 = S(\alpha\beta)^2 = 0, \quad S \,.\, \alpha^2\beta = 0, \quad S \,.\, \alpha\beta^2 = 0.$$

I shall presently show that the ten conditions

$$Sa = 0, \qquad S\alpha^2 = 0, \qquad S\alpha^3 = 1,$$

$$S\beta = 0, \qquad S\beta^2 = 0, \qquad S\beta^3 = 1,$$

$$S \,.\, \alpha\beta = 0, \quad S \,.\, \alpha^2\beta = 0, \quad S \,.\, \alpha\beta^2 = 0, \quad S(\alpha\beta)^2 = 0,$$

are sufficient as well as necessary. From

$$S\alpha = 0, \quad S\alpha^2 = 0, \quad S\alpha^3 = 1, \quad \text{we get} \quad \alpha^3 = 1\dagger;$$

and from

$$S\beta = 0, \quad S\beta^2 = 0, \quad S\beta^3 = 1, \quad \text{we get} \quad \beta^3 = 1.$$

Similarly,

$$S \,.\, \alpha\beta = 0, \quad S(\alpha\beta)^2 = 0, \quad \text{give} \quad (\alpha\beta)^3 - S(\alpha\beta)^3 = 0;$$

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* Sylvester remarks that "the solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism." Evidently the introduction into the theory of nonions, and into the theory of matrices in general, of the symbol S furnishes such an expansion of the idea of orthogonalism. Thus, just as in quaternions, two quaternion square roots of unity (*i.e.*, unit vectors) α and β are normal if $S\alpha\beta = 0$, when we have $\beta\alpha = -\alpha\beta$; so in nonions, two nonion cube roots of unity α, β may be regarded as completely normal, if $S\alpha\beta = 0$, $S(\alpha\beta)^2 = 0$, $S\alpha^2\beta = 0$, $S\alpha\beta^2 = 0$, when we have $\beta\alpha = \lambda\alpha\beta$, λ being an imaginary cube root of unity. It should be observed that the nonions α, β are not the most general forms of nonion unit vectors; the most general form is $xi + yi^2$, where i is a nonion cube root of unity, and $x^3 + y^3 = 1$.

† This is at once evident from the form of the identical equation of any nonion m,

$$m^3 - 3Sm \,.\, m^2 + \tfrac{3}{2}(3S^2m - Sm^2) \,.\, m + (\tfrac{9}{2}S^3m - \tfrac{9}{2}Sm \,.\, Sm^2 + Sm^3) = 0.$$

but $S(\alpha\beta)^3$ is here the explicit scalar term in the identical equation of $\alpha\beta$, and so is

$$\text{Det.}\,(\alpha\beta) = \text{Det.}\,(\alpha) \times \text{Det.}\,(\beta) = S\alpha^3 \,.\, S\beta^3 = 1\,;$$

hence $(\alpha\beta)^3 = 1$. We may now derive the equation $S\,.\,\alpha^2\beta^2 = 0$,[*] involved in the ten conditions, and necessary to the proof. For, since $(\alpha\beta)^3 = 1$, hence $\beta\alpha\,.\,\beta\alpha = \alpha^2(\alpha\beta)^3\beta^2 = \alpha^2\beta^2$; and consequently

$$S\,.\,\alpha^2\beta^2 = S(\beta\alpha)^2 = S(\alpha\beta)^2 = 0.$$

The proof that the ten conditions are sufficient depends upon the lemma that the involutant of α, β does not vanish.[†] This may be shown very simply as follows. If

$$[\alpha, \beta] \equiv (a_0 + a_1\alpha + a_2\alpha^2) + (b_0 + b_1\alpha + b_2\alpha^2)\,\beta + (c_0 + c_1\alpha + c_2\alpha^2)\,\beta^2 = 0,$$

then, since the scalar of all these products of powers of α and β is zero, $a_0 = S[\alpha, \beta] = 0$; and multiplying successively $[\alpha, \beta] = 0$ by α^2 and α, we get

$$a_1 = S\,.\,\alpha^2[\alpha, \beta] = 0, \quad \text{and} \quad a_2 = S\,.\,\alpha[\alpha, \beta] = 0;$$

similarly, multiplying $[\alpha, \beta]$ successively by 1, α^2, α, and into β^2, we get

$$b_0 = S.[\alpha, \beta]\,\beta^2 = 0, \quad b_1 = S.\alpha^2[\alpha, \beta]\,\beta^2 = 0, \quad b_2 = S.\alpha[\alpha, \beta]\,\beta^2 = 0.$$

In the same way we may show that the c's are zero.

Hence we may put

$$\beta\alpha = (x_0 + x_1\alpha + x_2\alpha^2) + (y_0 + y_1\alpha + y_2\alpha^2)\,\beta + (z_0 + z_1\alpha + z_2\alpha^2)\,\beta^2\,;$$

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[*] The equation $S\,.\,\alpha^2\beta^2 = 0$ may be substituted for the condition $S(\alpha\beta)^2 = 0$, which may be derived from it and seven of the other nine conditions. Thus

$$\beta\alpha\,.\,\beta\alpha = \alpha^2(\alpha\beta)^3\beta^2 = \alpha^2\left\{3S\,.\,\alpha\beta\,.\,(\alpha\beta)^2 - \tfrac{3}{2}\left(3S^2\,.\,\alpha\beta - S\,\overline{\alpha\beta}^2\right).\,\alpha\beta + S(\alpha\beta)^3\right\}\beta^2$$
$$= \tfrac{3}{2}\,S\,.\,(\alpha\beta)^2 + S\,.\,(\alpha\beta)^3\,.\,\alpha^2\beta^2,$$

if $\alpha^3 = \beta^3 = 1$ and $S\,.\,\alpha\beta = 0$. Hence, if $S\alpha^2\beta^2 = 0$,

$$S(\alpha\beta)^2 = S(\beta\alpha)^2 = \tfrac{3}{2}S(\alpha\beta)^2 + S(\alpha\beta)^3\,.\,S.\,\alpha^2\beta^2 = \tfrac{3}{2}S(\alpha\beta)^2\,;$$

i.e., $$S(\alpha\beta)^2 = 0.$$

[†] The *involutant* of m, n, two matrices of order ω, is the resultant of the ω^2 scalar equations obtained by equating to zero a linear function with scalar coefficients of the ω^2 matrices which result from multiplying $1, m, m^2, \ldots m^{\omega-1}$ into $1, n, n^2, \ldots n^{\omega-1}$. Obviously, if the involutant of (m, n) does not vanish, these ω^2 products are linearly independent, and any matrix of order ω may be expressed linearly in terms of them. In general, the m, n involutant (*i.e.*, the involutant of m, n) differs from the n, m involutant (*i.e.*, the involutant of n, m).

and, since $S\beta\alpha = 0$, hence $z_0 = 0$. Moreover,

$$z_1 = S\,.\,\alpha^2\,(\beta\alpha) = S\,.\,\alpha^3\beta = S\beta = 0,$$

$$z_2 = S\,.\,\alpha\;(\beta\alpha) = S\,.\,\alpha^2\beta = 0,$$

$$y_0 = S\,.\,(\beta\alpha)\,\beta^2 = S\,.\,\alpha\beta^3 = S\alpha = 0.$$

$$z_0 = S\,.\,(\beta\alpha)\,\beta = S\,.\,\alpha\beta^2 = 0,$$

$$z_2 = S\,.\,\alpha(\beta\alpha)\,\beta = S(\alpha\beta)^2 = 0\,;$$

therefore $$\beta\alpha = y_1\alpha\beta + y_2\alpha\beta^2 + z_1\alpha^2\beta\,;$$

therefore $$1 = (\beta\alpha)^3 = \beta\alpha\,(y_1\alpha\beta + y_2\alpha\beta^2 + z_1\alpha^2\beta)\,\beta\alpha$$
$$= y_1\beta\alpha^2\beta^2\alpha + y_2\beta + z_1\alpha\,;$$

therefore $$0 = S\beta^2 \;= y_1 S\beta^2 + y_2 + z_1 S\,.\,\alpha\beta^2 = y_2,$$
$$0 = S\alpha^2 \;= y_1 S_1\alpha^2 + y_2 S\,.\,\alpha^2\beta + z_1 = z_1\,;$$

therefore $$\beta\alpha = y_1\alpha\beta.$$

Since $(\alpha\beta)^3 = (\beta\alpha)^3 = 1$, y_1 is a cube root of unity; and since

$$(1 + y_1 + y_1^2)\,\alpha^2\beta = \alpha^2\beta + \alpha\beta\alpha + \beta\alpha^2 = 0,^{*}$$

hence y_1 is an imaginary cube root of unity.

If now i, j are Sylvester's special solutions (since the i, j involutant does not vanish), we may put for α, β (the matrices of the general solution),

$$\alpha = (a_0 + a_1 i + a_2 i^2) + (b_0 + b_1 i + b_2 i^2)\,j + (c_0 + c_1 i + c_2 i^2)\,j^2,$$

$$\beta = (a_0' + a_1' i + a_2' i^2) + (b_0' + b_1' i + b_2' i^2)\,j + (c_0' + c_1' i + c_2' i^2)\,j^2\,;$$

and if we write

$$\alpha^2 = (A_0 + A_1\,i + A_2\,i^2) + (B_0 + B_1\,i + B_2\,i^2)\,j + (C_0 + C_1\,i + C_2\,i^2)\,j^2,$$

$$\beta^2 = (A_0' + A_1'\,i + A_2'\,i^2) + (B_0' + B_1'\,i + B_2'\,i^2)\,j + (C_0' + C_1'\,i + C_2'\,i^2)\,j^2,$$

$$\alpha\beta = (A_0'' + A_1'' i + A_2'' i^2) + (B_0'' + B_1'' i + B_2'' i^2)\,j + (C_0'' + C_1'' i + C_2'' i^2)\,j^2,$$

---

[*] This follows immediately from the form of the second equation in the catena α, β, namely,

$$\begin{aligned} 0 = (\alpha^2\beta + \alpha\beta\alpha + \beta\alpha^2) - 3S\beta\,.\,\alpha^2 - 3S\alpha\,.\,(\alpha\beta + \beta\alpha) \\ + 3\,(3S\alpha\,.\,S\beta - S\,.\,\alpha\beta)\,.\,\alpha + \tfrac{3}{2}\,(3S^2\alpha - S\alpha^2)\,.\,\beta \\ - (\tfrac{27}{2} S^2\alpha\,.\,S\beta - \tfrac{9}{2} S\beta\,.\,S\alpha^2 - 9S\alpha\,.\,S\,.\,\alpha\beta + 3S\,.\,\alpha^2\beta) \end{aligned}$$
$$= \alpha^2\beta + \alpha\beta\alpha + \beta\alpha^2.$$

then the necessary and sufficient conditions expressed in terms of these coefficients that α and β shall also be solutions, are

$$a_0 = S\alpha = 0, \quad A_0 = S\alpha^2 = 0,$$

$$b_0 = S\beta = 0, \quad B_0 = S\beta^2 = 0,$$

$$a_0 a_0' + (a_1 a_2' + a_1' a_2) + (b_0 c_0' + b_0' c_0) + \lambda^2 (b_1 c_2' + b_1' c_2)$$
$$+ \lambda (b_2 c_1' + b_2' c_1) = S \,.\, \alpha\beta = 0,$$

$$A_0 a_0' + (A_1 a_2' + a_1' A_2) + (B_0 c_0' + b_0' C_0) + \lambda^2 (B_1 c_2' + b_1' C_2)$$
$$+ \lambda (B_2 c_1' + b_2' C_1) = S \,.\, \alpha^2\beta = 0,$$

$$a_0 A_0' + (a_1 A_2' + A_1' a_2) + (b_0 C_0' + B_0' c_0) + \lambda^2 (b_1 C_2' + B_1' c_2)$$
$$+ \lambda (b_2 C_1' + B_2' c_1) = S \,.\, \alpha\beta^2 = 0,$$

$$a_0 A_0 + (a_1 A_2 + a_2 A_1) + (b_0 C_0 + B_0 c_0) + \lambda^2 (b_1 C_2 + B_1 c_2)$$
$$+ \lambda (b_2 C_1 + B_2 c_1) = S \,.\, \alpha^3 = 1,$$

$$a_0' A_0' + (a_1' A_2' + a_2' A_1') + (b_0' C_0' + B_0' c_0') + \lambda^2 (b_1' C_2' + B_1' c_2')$$
$$+ \lambda (b_2' C_1' + B_2' c_1') = S \,.\, \beta^3 = 1,$$

$$A_0'' A_0'' + 2A_1'' A_2'' + 2B_0'' C_0'' + 2\lambda^2 B_1'' C_2'' + \lambda B_2'' C_1'' = S (\alpha\beta)^2 = 0.$$

If the latent function of the corpus m, n is

Det. $(x + ym + zn)$

$$= x^3 + 3Bx^2y + 3Cx^2z + 3Dxy^2 + 6Exyz + 3Fxz^2 + Gy^3 + 3Hy^2z$$
$$+ 3Kyz^2 + Lz^3,$$

then $\quad B = Sm, \qquad C = Sn,$

$$D = \tfrac{1}{2} (3S^2m - Sm^2), \quad E = \tfrac{1}{2} (3Sm \,.\, Sn - S \,.\, mn),$$

$$F = \tfrac{1}{2} (3S^2m - Sn^2), \quad G = \tfrac{9}{2} S^3m - \tfrac{9}{2} Sm \,.\, Sm^2 + Sm^3,$$

$$H = \tfrac{9}{2} S^2m \,.\, Sn - \tfrac{3}{2} Sn \,.\, Sm^2 - 3Sm \,.\, S \,.\, mn + S \,.\, m^2n,$$

$$K = \tfrac{9}{2} Sm \,.\, S^2n - 3Sn \,.\, S \,.\, mn - \tfrac{3}{2} Sm \,.\, Sn^2 + S \,.\, mn^2,$$

$$L = \tfrac{9}{2} S^3n - \tfrac{9}{2} Sn \,.\, Sn^2 + Sn^3.$$

As given by Sylvester, the first 9 conditions are

$$B = 0, \quad C = 0, \quad D = 0, \quad E = 0, \quad F = 0, \quad H = 0, \quad K = 0,$$

and $$G = 1, \quad L = 1.$$

This gives $$\text{Det.}\,(x+ym+zn) = x^3+y^3+z^3.$$

It is evident, from the above values of the coefficients of the latent function, that these conditions are equivalent to 9 of the conditions as given by me. Sylvester's 10th condition is that the coefficient of mn in the identical equation of mn shall vanish. This coefficient is, however, $\frac{3}{2}\left[3S^2 \,.\, mn - S\,(mn)^2\right]$; and, as $S \,.\, mn = 0$ (which follows from $B = 0$, $C = 0$, $E = 0$), Sylvester's 10th condition reduces to my 10th condition, namely,

$$S \,.\, (mn)^2 = 0.$$

4. *Expression for the involutants* of two matrices in terms of the sums of the latent roots of products of powers of the matrices.*

For simplicity, I consider the involutants of the two quaternions q and r; but the form of the involutants for matrices of higher order is at once evident, and the same demonstration applies in any case. If the q, r involutant vanishes, then

$$Q \equiv (a+bq)+(c+dq)\,r = 0,$$

for some value of the scalars a, b, c, d. Therefore

$$\left.\begin{array}{ll} SQ = 0, & S\,.\,qQ = 0 \\ S\,.\,rQ = 0, & S\,.\,qrQ = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

The resultant of equations (A) is

$$I \equiv \begin{vmatrix} 1, & Sq, & Sr, & S\,.\,qr \\ Sq, & Sq^2, & S\,.\,qr, & S\,.\,q\,(qr) \\ Sr, & S\,.\,rq, & Sr^2, & S\,.\,r\,(qr) \\ S\,.\,qr, & S\,.\,(qr)\,q, & S\,.\,(qr)\,r, & S\,.\,(qr)^2 \end{vmatrix}.$$

I is, of course, a multiple of the q, r involutant.

It is readily seen that the equations (A) do not all coexist, unless $Q = 0$. For, otherwise, from equations (A), we obtain

$$S\left[\,\{x+yq+zr+w\,(qr)\}\,Q\right] \equiv 0$$

---

* See note †, p. 73.

for any values of the scalars x, y, z, w; and, if $Q \neq 0$, $x+yq+zr+w\,(qr)$ may be any quaternion; hence

$$S\,(RQ) = 0$$

for any quaternion R, which is impossible unless

$$Q = 0.$$

For let the two quaternions i, j be such that

$$i^2 = j^2 = 1, \quad ji = -ij\,;$$

then the involutant of i, j does not vanish.* Let the expression for Q in terms of $(1, i)(1, j)$ be

$$Q = (A+Bi)+(C+Di)\,j\,;$$

then $\quad A = SQ = 0, \qquad\qquad B = S\,.\,iQ = 0,$

$$C = S\,.\,Qj = S\,.\,jQ = 0, \quad D = S\,.\,iQj = S\,(ji)\,Q = 0.$$

Hence I vanishes, if and only if $Q = 0$; *and consequently I is some power of the involutant of q, r.*

The involutant of r, q is similarly the resultant of the equations

$$SQ' = 0, \quad S\,.\,rQ' = 0, \quad S\,.\,qQ' = 0, \quad S\,.\,rqQ' = 0,$$

where $\qquad Q' = (a'+b'r)+(c'+d'r)\,q,$

a', b', c', d' being scalars. If J denotes this resultant, then

$$J \equiv \begin{vmatrix} 1, & Sr, & Sq, & S\,.\,qr \\ Sr, & Sr^2, & S\,.\,rq, & S\,.\,r\,(qr) \\ Sq, & S\,.\,qr, & Sq^2, & S\,.\,q\,(qr) \\ S\,.\,rq, & S\,.\,(rq)\,r, & S\,.\,(rq)\,q, & S\,.\,(qr)^2 \end{vmatrix}.$$

---

* The same method employed in § 3 to show that the involutant of i, j does not vanish for the two nonions i and j, for which

$$Si = Si^2 = 0, \quad Sj = Sj^2 = 0, \quad S\,.\,ij = S\,.\,i^2j = S\,.\,ij^2 = 0,$$

may be applied to show that the i, j involutant of two matrices i, j of order ω does not vanish, if

$$S\,.\,i^p j^q = 0,$$

for all values of p and q from 0 to $\omega-1$, except $p = q = 0$.

And, since $S\,.\,qr = S\,.\,rq,\quad S\,.\,r\,(qr) = S\,.\,r\,(rq)$, &c.,

hence $$I = J.$$

It is of interest to observe that the involutant of the two quaternions vectors α, β is

$$\left\{S\,(\alpha\beta)^2 - S^2\,.\,\alpha\beta\right\}(\alpha^2\beta^2 - S^2\,.\,\alpha\beta) = V^4\,.\,\alpha\beta = T^4V\,.\,\alpha\beta.$$

It is, of course, otherwise evident that the involutant must be some multiple of a power of $TV\alpha\beta$.

The m, n involutant of the two nonions m and n is the resultant of the equations

$$\begin{aligned} SM &= 0, & S\,.\,mM &= 0, & S\,.\,m^2M &= 0, \\ S\,.\,nM &= 0, & S\,.\,mnM &= 0, & S\,.\,m^2nM &= 0, \\ S\,.\,n^2M &= 0, & S\,.\,mn^2M &= 0, & S\,.\,m^2n^2M &= 0, \end{aligned}$$

where $$M = (a_0, a_1, a_2, \ldots a_8 \between 1, m, m^2 \between 1, n, n^2);$$

and the n, m involutant is the resultant of the equations

$$\begin{aligned} SM' &= 0, & S\,.\,mM' &= 0, & S\,.\,m^2M' &= 0, \\ S\,.\,nM' &= 0, & S\,.\,nmM' &= 0, & S\,.\,nm^2M' &= 0, \\ S\,.\,n^2M' &= 0, & S\,.\,n^2mM' &= 0, & S\,.\,n^2m^2M' &= 0, \end{aligned}$$

where $$M' = (a'_0, a'_1, a'_2, \ldots a'_8 \between 1, n, n^2 \between 1, m, m^2).$$

Both these involutants are also symmetric matrices, and so in general. In general, the m, n involutant I of two matrices, m and n, of the third or higher order, is not equal to the n, m involutant J of the matrices. For the two nonions m and n, I find

$$S\,(m^2n^2mn) = S\,(n^2m^2nm)$$

to be a sufficient condition that $I = J$; for, then, on comparing the determinants I and J, it will be found that every constituent of the one is equal to the corresponding constituent of the other.

5. *Proof of the theorem that the latent roots of mn and nm are identical.*

This theorem has been given by Sylvester (*Johns Hopkins Circs.*, Vol. III., p. 8) for matrices in general, and earlier by Tait (*Elements*, Chap. v., p. 103), and probably by Hamilton, for matrices of the third order.

It is, of course, only necessary to show that the sum of the k^{th} powers of the latent roots of mn [*i.e.*, the sum of the latent roots of $(mn)^k$] is equal to the sum of the k^{th} powers of the latent roots of nm [*i.e.*, the sum of the latent roots of $(nm)^k$]. But

$$S\,(mn)^k = S\,.\,m\,(nm)^{k-1}\,n = S\,.\,nm\,(nm)^{k-1} = S\,(nm)^k.$$

Note.—By an oversight the word "involutant" has been used above in § 4 for "square of the involutant." Thus, the above resultants I and J are the *squares* of the corresponding involutants.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables. By E. B. Elliott.

[*Read Dec.* 11*th*, 1890.]

1. The theory of the interchange of the dependent and independent variables, in functions of the derivatives of one variable with regard to another of which it is by supposition a function, has of late years been studied with great elaboration on the basis of Prof. Sylvester's new discovery of reciprocants. The last five volumes of our *Proceedings* abound in valuable contributions to the study.

Mr. A. Berry, in his paper on "Simultaneous Reciprocants" (*Quarterly Journal*, 1888, p. 260), has initiated an analogous theory as to the reversion, by interchange of x and y and of x' and y', of functions of the two sets of ordinary derivatives

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ldots\ \frac{dy'}{dx'},\ \frac{d^2y'}{dx'^2},\ \frac{d^3y'}{dx'^3},\ldots;$$

his supposition being that x, y are connected by one relation, and x', y' by another.

It would appear that the theory of the interchange of the dependent and independent pairs in functions of the partial derivatives of two variables with regard to two others, the four being connected by two relations of perfectly arbitrary form, has not hitherto been dealt with in the light of recent researches. It is on the development of this theory that I now propose to embark.

2. Let x, y be two variables connected with two others x', y' by two relations

$$f_1(x, y, x', y') = 0, \quad f_2(x, y, x', y') = 0,$$

whose form is immaterial and need not be known. We may regard either x, y as independent and x', y' as functions of them, or x', y' as independent and x, y as dependent. It is required to investigate connexions between the various partial derivatives in the one case, and in the other. Two important branches of the investigation will be the search for the invariants or reciprocants of the reversion,—*i.e.*, functions of the derivatives which have the same form, but for a simple factor in the two cases,—and the consideration of the reversion of operators linear in symbols of partial differentiation with regard to the partial derivatives.

Let us once for all denote, for zero and positive integral values of r and s,

$$\frac{1}{r!\,s!}\,\frac{d^{r+s}x}{dx'^r dy'^s}, \quad \frac{1}{r!\,s!}\,\frac{d^{r+s}y}{dx'^r dy'^s},$$

$$\frac{1}{r!\,s!}\,\frac{d^{r+s}x'}{dx^r dy^s}, \quad \frac{1}{r!\,s!}\,\frac{d^{r+s}y'}{dx^r dy^s},$$

by x_{rs}, y_{rs}, x'_{rs}, y'_{rs} respectively.

The values of either pair of first derivatives in terms of the other pair follow at once by a well-known method from the identity of the pair of equalities

$$dx = x_{10}dx' + x_{01}dy',$$

$$dy = y_{10}dx' + y_{01}dy',$$

with the pair

$$dx' = x'_{10}dx + x'_{01}dy,$$

$$dy' = y'_{10}dx + y'_{01}dy.$$

We are thus given that

$$\left.\begin{aligned} x_{10}x'_{10} + x_{01}y'_{10} &= 1 \\ x_{10}x'_{01} + x_{01}y'_{01} &= 0 \\ y_{10}x'_{10} + y_{01}y'_{10} &= 0 \\ y_{10}x'_{01} + y_{01}y'_{01} &= 1 \end{aligned}\right\} \quad \text{..........................} \quad (1),$$

so that

$$\frac{x_{10}}{y'_{01}} = -\frac{x_{01}}{x'_{01}} = -\frac{y_{10}}{y'_{10}} = \frac{y_{01}}{x'_{10}} = x_{10}y_{01} - x_{01}y_{10} = \frac{1}{x'_{10}y'_{01} - x'_{01}y'_{10}} \quad \text{......(2).}$$

The successive derivation of expressions for higher derivatives x_{mn}, y_{mn} is, by means of the operator equivalents,

$$\left.\begin{aligned} \frac{d}{dx} &= (x_{10}y_{01}-x_{01}y_{10})^{-1}\left\{ \quad y_{01}\frac{d}{dx'} - y_{10}\frac{d}{dy'}\right\} \\ \frac{d}{dy} &= (x_{10}y_{01}-x_{01}y_{10})^{-1}\left\{ -x_{01}\frac{d}{dx'} + x_{10}\frac{d}{dy'}\right\} \end{aligned}\right\} \quad \text{..............(3),}$$

which follow from the facts,

$$\left.\begin{aligned} \frac{du}{dx} &= \frac{d(u, y)}{d(x, y)} = \left\{\frac{d(x, y)}{d(x', y')}\right\}^{-1} \frac{d(u, y)}{d(x', y')} \\ \frac{du}{dy} &= -\frac{d(u, x)}{d(x, y)} = -\left\{\frac{d(x, y)}{d(x', y')}\right\}^{-1} \frac{d(u, x)}{d(x', y')} \end{aligned}\right\} \quad \text{......... (4).}$$

In this way, for instance, we obtain in the next place that

$$\begin{aligned} x'_{20} = {} & (x_{10}y_{01}-x_{01}y_{10})^{-3} \\ & \times\left\{x_{01}\left(y_{01}^2y_{20}-y_{01}y_{10}y_{11}+y_{10}^2y_{02}\right)-y_{01}\left(y_{01}^2x_{20}-y_{01}y_{10}x_{11}+y_{10}^2x_{02}\right)\right\} \\ & \text{.....................(5),} \end{aligned}$$

$$\begin{aligned} x'_{11} = {} & -(x_{10}y_{01}-x_{01}y_{10})^{-3}\left\{x_{01}\left(2y_{01}x_{01}y_{20}-\overline{y_{01}x_{10}+y_{10}x_{01}}\,y_{11}+2y_{10}x_{10}y_{02}\right)\right. \\ & \left.-y_{01}\left(2y_{01}x_{01}x_{20}-\overline{y_{01}x_{10}+y_{10}x_{01}}\,x_{11}+2y_{10}x_{10}x_{02}\right)\right\} \text{..............(6),} \end{aligned}$$

$$\begin{aligned} x'_{02} = {} & (x_{10}y_{01}-x_{01}y_{10})^{-3} \\ & \times\left\{x_{01}\left(x_{01}^2y_{20}-x_{01}x_{10}y_{11}+x_{10}^2y_{02}\right)-y_{01}\left(x_{01}^2x_{20}-x_{01}x_{10}x_{11}+x_{10}^2x_{02}\right)\right\} \\ & \text{.....................(7),} \end{aligned}$$

$$\begin{aligned} y'_{20} = {} & (x_{10}y_{01}-x_{01}y_{10})^{-3} \\ & \times\left\{-x_{10}\left(y_{01}^2y_{20}-y_{01}y_{10}y_{11}+y_{10}^2y_{02}\right)+y_{10}\left(y_{01}^2x_{20}-y_{01}y_{10}x_{11}+y_{10}^2x_{02}\right)\right\} \\ & \text{.....................(8),} \end{aligned}$$

$$\begin{aligned} y'_{11} = {} & -(x_{10}y_{01}-x_{01}y_{10})^{-3}\left\{-x_{10}\left(2y_{01}x_{01}y_{20}-\overline{y_{01}x_{10}+y_{10}x_{01}}\,y_{11}+2y_{10}x_{10}y_{02}\right)\right. \\ & \left.+y_{10}\left(2y_{01}x_{01}x_{20}-\overline{y_{01}x_{10}+y_{10}x_{01}}\,x_{11}+2y_{10}x_{10}x_{02}\right)\right\} \text{...... (9),} \end{aligned}$$

$$\begin{aligned} y'_{02} = {} & (x_{10}y_{01}-x_{01}y_{10})^{-3} \\ & \times\left\{-x_{10}\left(x_{01}^2y_{20}-x_{01}x_{10}y_{11}+x_{10}^2y_{02}\right)+y_{10}\left(x_{01}^2x_{20}-x_{01}x_{10}x_{11}+x_{10}^2x_{02}\right)\right\} \\ & \text{.....................(10).} \end{aligned}$$

The general formulæ are

$$\left.\begin{aligned}(m+1)\,!\;n\,!\;x'_{m+1\,n} &= \partial^m\partial'^n\left\{(x_{10}y_{01}-x_{01}y_{10})^{-1}y_{01}\right\}\\ m\,!\;(n+1)\,!\;x'_{m\,n+1} &= \partial^m\partial'^n\left\{-(x_{10}y_{01}-x_{01}y_{10})^{-1}x_{01}\right\}\\ (m+1)\,!\;n\,!\;y'_{m+1\,n} &= \partial^m\partial'^n\left\{-(x_{10}y_{01}-x_{01}y_{10})^{-1}y_{10}\right\}\\ m\,!\;(n+1)\,!\;y'_{m\,n+1} &= \partial^m\partial'^n\left\{(x_{10}y_{01}-x_{01}y_{10})^{-1}x_{10}\right\}\end{aligned}\right\}\text{......(11),}$$

where ∂, ∂' are the two dexters of the equivalences (3). In these identities (11), m and n may both or either be zero.

3. A function of the unaccented derivatives whose equivalent in terms of the accented is of exactly the same form, but for a sign factor at most, is, in accordance with the nomenclature of analogous theories, called an *absolute reciprocant*. A function whose equivalent is of the same form, but for a power of the Jacobian

$$x'_{10}y'_{01}-x'_{01}y'_{10}$$

as factor, and also it may be by a sign factor, is called a *reciprocant*. If there be a negative sign factor, a reciprocant is said to be of negative character; otherwise, of positive.

From the identities (2) we have at once the means of writing down five fundamental absolute reciprocants of the first order, namely,

$$\alpha \equiv \log\,(x_{10}y_{01}-x_{01}y_{10})\;\text{.............................}\;(12),$$

$$\beta \equiv \quad (x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}x_{01}\;\text{........................}\;(13),$$

$$\gamma \equiv \quad (x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}y_{10}\;\text{........................}\;(14),$$

$$\delta \equiv \quad (x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}(x_{10}-y_{01})\;\text{..............}\;(15),$$

$$\epsilon \equiv \quad (x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}(x_{10}+y_{01})\;\text{..............}\;(16),$$

whose characters, respectively, are

$$-,\quad -,\quad -,\quad -,\quad +.$$

The last four are connected by a relation, namely,

$$\epsilon^2-\delta^2-4\beta\gamma = 4\text{..............................}(17).$$

Thus, in one sense, the four of negative character form a complete system of the first order. There must, however, be two not explicitly known reciprocants whose Jacobian, namely,

$$(x_{10}y_{01}-x_{01}y_{10})^{\frac{1}{2}}\text{....... ....}(18),$$

is of the first order, to which order they must themselves be regarded as prior, depending at any rate partly on the variables themselves, and not on their derivatives only. Their product is

$$\iint (x_{10}y_{01}-x_{01}y_{10})^{\frac{1}{2}}\,dx'dy' = \iint (dx\,dy\,dx'dy')^{\frac{1}{2}}$$

$$= \iint (x'_{10}y'_{01}-x'_{01}y'_{10})^{\frac{1}{2}}\,dx\,dy \quad \ldots\ldots\ldots\ldots \quad (19).$$

Their importance undoubtedly underlies the theory of the reciprocants β, γ, δ, ϵ, and all that can be deduced from them, just as the ordinary reciprocant, $\int \sqrt{t}\,dx$, is of capital importance in Professor Sylvester's theory.

4. The following is the central theorem as to the eduction of reciprocants of higher orders:—"If u, v, w are three absolute reciprocants, then $\dfrac{d^{m+n}u}{dv^m\,dw^n}$ is another, m and n having any positive integral or zero values."

This is clear, for $\dfrac{d}{dv}$ and $\dfrac{d}{dw}$ must be reciprocant operators; or again, because $\dfrac{1}{m!\,n!}\,\dfrac{d^{m+n}u}{dv^m\,dw^n}$ is the general coefficient in the expansion of an increment of u in terms of increments of v and w.

Accordingly, since, as in (4),

$$\frac{d\phi}{dv} = \frac{d(\phi, w)}{d(x', y')}\left\{\frac{d(v, w)}{d(x', y')}\right\}^{-1},$$

and

$$\frac{d\phi}{dw} = -\frac{d(\phi, v)}{d(x', y')}\left\{\frac{d(v, w)}{d(x', y')}\right\}^{-1},$$

it follows that

$$\left\{\left(\frac{dv}{dx'}\frac{dw}{dy'}-\frac{dw}{dx'}\frac{dv}{dy'}\right)^{-1}\left(\frac{dw}{dy'}\frac{d}{dx'}-\frac{dw}{dx'}\frac{d}{dy'}\right)\right\}^{m}$$

$$\times\left\{\left(\frac{dv}{dx'}\frac{dw}{dy'}-\frac{dw}{dx'}\frac{dv}{dy'}\right)^{-1}\left(\frac{dv}{dy'}\frac{d}{dx'}-\frac{dv}{dx'}\frac{d}{dy'}\right)\right\}^{n}u \quad \ldots\ldots\ldots\ldots \quad (20)$$

is the type of an infinite series of absolute reciprocants educed from u by means of v and w.

If $(-1)^{k_1}$, $(-1)^{k_2}$, $(-1)^{k_3}$ be the characters of u, v, w, the character of $\dfrac{d^{m+n}u}{dv^m\,dw^n}$ is $(-1)^{mk_2+nk_3-k_1}$.

5. A simpler form may be given to the aggregate (20) by aid of the theorem that the Jacobian of two absolute reciprocants is a reciprocant. This is clear, for

$$\frac{d(u,v)}{d(x,y)}=\frac{d(x',y')}{d(x,y)}\,\frac{d(u,v)}{d(x',y')};$$

so that, remembering the last equality in (2),

$$(x'_{10}y'_{01}-x'_{01}y'_{10})^{-\frac{1}{2}}\frac{d(u,v)}{d(x,y)}=(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}\frac{d(u,v)}{d(x',y')}\text{ ....} \quad (21).$$

We may express this by saying that the Jacobian of any two absolute reciprocants is a reciprocant of *index* $\frac{1}{2}$, the index being that power of $x_{10}y_{01}-x_{01}y_{10}$ by which a reciprocant has to be divided to make it absolute.

We have, hence, as the type of a system of absolute reciprocants, equivalent in the aggregate to (20),

$$\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}\left(\frac{dw}{dy'}\frac{d}{dx'}-\frac{dw}{dx'}\frac{d}{dy'}\right)\right\}^{m}$$

$$\times\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}\left(\frac{dv}{dy'}\frac{d}{dx'}-\frac{dv}{dx'}\frac{d}{dy'}\right)\right\}^{n}u\text{ ....} (22).$$

Again, we can use the absolute reciprocants (13) to (16) to produce further apparent simplification. Thus, for instance, we may replace, in (22),

$$(x_{10}y_{01}-x_{01}y_{10})^{\frac{1}{2}}$$

by $\quad x_{01},\quad y_{10},\quad x_{10}-y_{01}\quad$ or $\quad x_{10}+y_{01}$.

There will, it is naturally to be expected, be *pure* reciprocants, *i.e.*, reciprocants in which the first derivatives are absent and higher ones alone occur. Theorems as to such will best follow the investigation of the reversion of linear differential operators, to which we next proceed.

6. The object now before us is to express such an operator as

$$P_{10}\frac{d}{dx_{10}}+P_{01}\frac{d}{dx_{01}}+P_{20}\frac{d}{dx_{20}}+\ldots+Q_{10}\frac{d}{dy_{10}}+Q_{01}\frac{d}{dy_{01}}+Q_{20}\frac{d}{dy_{20}}+\ldots,$$

supposed to act on any function of the unaccented derivatives, as a sum of multiples of the symbols

$$\frac{d}{dx'_{10}},\ \frac{d}{dx'_{01}},\ \frac{d}{dx'_{20}},\ \ldots\ \frac{d}{dy'_{10}},\ \frac{d}{dy'_{01}},\ \frac{d}{dy'_{20}},\ \ldots,$$

operating on the equivalent function of the accented derivatives.

Let ξ, η, ξ', η' be possible simultaneous finite increments of x, y, x', y'. The pair of relations connecting them may, by Taylor's theorem, be given the forms

$$\xi' = x'_{10}\xi + x'_{01}\eta + x'_{20}\xi^2 + x'_{11}\xi\eta + x'_{02}\eta^2 + \dots \quad \dots\dots\dots\dots \quad (23),$$

$$\eta' = y'_{10}\xi + y'_{01}\eta + y'_{20}\xi^2 + y'_{11}\xi\eta + y'_{02}\eta^2 + \dots \quad \dots\dots\dots\dots \quad (24),$$

or, again, the forms

$$\xi = x_{10}\xi' + x_{01}\eta' + x_{20}\xi'^2 + x_{11}\xi'\eta' + x_{02}\eta'^2 + \dots \quad \dots\dots\dots\dots \quad (25),$$

$$\eta = y_{10}\xi' + y_{01}\eta' + y_{20}\xi'^2 + y_{11}\xi'\eta' + y_{02}\eta'^2 + \dots \quad \dots\dots\dots\dots \quad (26).$$

Now, the forms of the relations

$$f_1(x, y, x', y') = 0, \quad f_2(x, y, x', y') = 0,$$

which connect x, y, x', y', being arbitrary, we may, if we please, look upon

$$\xi, \eta, \quad x'_{10}, x'_{01}, x'_{20}, \dots y'_{10}, y'_{01}, y'_{20}, \dots$$

as independent quantities,

and upon $\quad \xi', \eta', \quad x_{10}, x_{01}, x_{20}, \dots y_{10}, y_{01}, y_{20}, \dots$

as quantities dependent on them, and determinate in terms of them; the first two by (23) and (24), and the rest by the equivalence of the relations (25) and (26) with (23) and (24).

Of the quantities

$$x'_{10}, x'_{01}, \dots y'_{10}, y'_{01}, \dots$$

keep, then, all constant except x'_{rs}, to which give an infinitesimal variation $\delta x'_{rs}$. Moreover, keep ξ and η constant. By (23) and (24), we get under these circumstances

$$\delta\xi' = \xi^r \eta^s \delta x'_{rs} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (27),$$

$$\delta\eta' = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (28);$$

and by (25) and (26),

$$\begin{aligned} 0 = & \left\{ \frac{dx_{10}}{dx'_{rs}}\xi' + \frac{dx_{01}}{dx'_{rs}}\eta' + \frac{dx_{20}}{dx'_{rs}}\xi'^2 + \frac{dx_{11}}{dx'_{rs}}\xi'\eta' + \frac{dx_{02}}{dx'_{rs}}\eta'^2 + \dots \right\} \delta x'_{rs} \\ & + \left\{ x_{10} + 2x_{20}\xi' + x_{11}\eta' + 3x_{30}\xi'^2 + 2x_{21}\xi'\eta' + x_{12}\eta'^2 + \dots \right\} \delta\xi' \\ & + \left\{ x_{01} + x_{11}\xi' + 2x_{02}\eta' + x_{21}\xi'^2 + 2x_{12}\xi'\eta' + 3x_{03}\eta'^2 + \dots \right\} \delta\eta' \\ & \dots\dots\dots\dots\dots\dots (29);\end{aligned}$$

$$
\begin{aligned}
0 = &\left\{ \frac{dy_{10}}{dx'_{rs}}\xi' + \frac{dy_{01}}{dx'_{rs}}\eta' + \frac{dy_{20}}{dx'_{rs}}\xi'^2 + \frac{dy_{11}}{dx'_{rs}}\xi'\eta' + \frac{dy_{02}}{dx'_{rs}}\eta'^2 + \dots \right\} \delta x'_{rs} \\
&+ \left\{ y_{10} \quad + 2y_{20}\xi' + y_{11}\eta' \quad + 3y_{30}\xi'^2 + 2y_{21}\xi'\eta' + y_{02}\eta'^2 \quad + \dots \right\} \delta\xi' \\
&+ \left\{ \quad y_{01} \quad + y_{11}\xi' + 2y_{02}\eta' \quad + y_{21}\xi'^2 + 2y_{12}\xi'\eta' + 3y_{03}\eta'^2 + \dots \right\} \delta\eta' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dots\dots\dots\dots\dots\dots (30).
\end{aligned}
$$

The two pairs of conditions must be identical. Expressing this identity, we obtain the equivalences

$$
\begin{aligned}
&\frac{dx_{10}}{dx'_{rs}}\xi' + \frac{dx_{01}}{dx'_{rs}}\eta' + \frac{dx_{20}}{dx'_{rs}}\xi'^2 + \frac{dx_{11}}{dx'_{rs}}\xi'\eta' + \frac{dx_{02}}{dx'_{rs}}\eta'^2 + \dots \\
&\qquad = -\left\{ x_{10} + 2x_{20}\xi' + x_{11}\eta' + 3x_{30}\xi'^2 + 2x_{21}\xi'\eta' + x_{12}\eta'^2 + \dots \right\} \xi^r\eta^s \\
&\qquad = -\frac{d\xi}{d\xi'}\xi^r\eta^s \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (31);
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{dy_{10}}{dx'_{rs}}\xi' + \frac{dy_{01}}{dx'_{rs}}\eta' + \frac{dy_{20}}{dx'_{rs}}\xi'^2 + \frac{dy_{11}}{dx'_{rs}}\xi'\eta' + \frac{dy_{02}}{dx'_{rs}}\eta'^2 + \dots \\
&\qquad = -\left\{ y_{10} + 2y_{20}\xi' + y_{11}\eta' + 3y_{30}\xi'^2 + 2y_{21}\xi'\eta' + y_{12}\eta'^2 + \dots \right\} \xi^r\eta^s \\
&\qquad = -\frac{d\eta}{d\xi'}\xi^r\eta^s \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (32),
\end{aligned}
$$

in which $\frac{d}{d\xi'}$ is indicative of strictly partial differentiation of the expressions for ξ, η in (25) and (26).

If to ξ and η, in the right of these equalities (31) and (32), we give their expanded values from (25) and (26), and if we then expand and arrange by powers and products of ξ' and η', the multipliers of products obtained must of course be equal to those of the same products in the left-hand members of the equalities. Thus, we conclude that, for any m and n,

$$
\begin{aligned}
&\frac{dx_{mn}}{dx'_{rs}} = \text{co. } \xi'^m\eta'^n \text{ in expansion of} \\
&-\frac{1}{r+1}\left\{ y_{10}\xi' + y_{01}\eta' + 2y_{20}\xi'^2 + \dots \right\}^s \frac{d}{d\xi'}\left\{ x_{10}\xi' + x_{01}\eta' + 2x_{20}\xi'^2 + \dots \right\}^{r+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dots\dots\dots\dots\dots\dots (33),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{dy_{mn}}{dx'_{rs}} = \text{co. } \xi'^m\eta'^n \text{ in expansion of} \\
&-\frac{1}{s+1}\left\{ x_{10}\xi' + x_{01}\eta' + 2x_{20}\xi'^2 + \dots \right\}^r \frac{d}{d\xi'}\left\{ y_{10}\xi' + y_{01}\eta' + 2y_{20}\xi'^2 + \dots \right\}^{s+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dots\dots\dots\dots\dots\dots (34).
\end{aligned}
$$

We have thus the means of obtaining the partial differential coefficient with regard to x'_{rs} of any function of the unaccented derivatives. The conclusion is thrown into a very simple form, as follows. Add (31) to λ times (32), where λ is perfectly arbitrary. On the left we thus get an expression in which, on substitution for

$$\xi',\ \lambda\xi',\ \eta',\ \lambda\eta',\ \xi'^2,\ \lambda\xi'^2,\ \xi'\eta',\ \lambda\xi'\eta',\ \dots\ \xi'^m\eta'^n,\ \lambda\xi'^m\eta'^n,\ \dots$$

of
$$\frac{d}{dx_{10}},\ \frac{d}{dy_{10}},\ \frac{d}{dx_{01}},\ \frac{d}{dy_{01}},\ \frac{d}{dx_{20}},\ \frac{d}{dy_{20}},\ \frac{d}{dx_{11}},\ \frac{d}{dy_{11}},\ \dots\ \frac{d}{dx_{mn}},\ \frac{d}{dy_{mn}},\ \dots,$$

we obtain

$$\frac{dx_{10}}{dx'_{rs}}\frac{d}{dx_{10}}+\frac{dy_{10}}{dx'_{rs}}\frac{d}{dy_{10}}+\frac{dx_{01}}{dx'_{rs}}\frac{d}{dx_{01}}+\frac{dy_{01}}{dx'_{rs}}\frac{d}{dy_{01}}+\dots,$$

which is the equivalent of the operative symbol $\frac{d}{dx'_{rs}}$. The same, then, must be the case on the right. In other words, we obtain the equivalence, in which on the left the operation is upon a function of the accented, and on the right upon the equal function of the accented variables,

$$\frac{d}{dx'_{rs}}=-\xi^r\eta^s\left\{\frac{d\xi}{d\xi'}+\lambda\frac{d\eta}{d\xi'}\right\}\ \dots\dots\dots\dots\dots\dots\dots(35),$$

the notation on the right being symbolical, and the meaning being that by aid of (25) and (26) the expansion as a sum of multiples of products of powers of ξ' and η' is to be effected, and that then, for every product, $\xi'^m\eta'^n$, as it occurs linearly without a λ factor, is to be written $\frac{d}{dx_{mn}}$, while, for every product, $\lambda\xi'^m\eta'^n$ is to be written $\frac{d}{dy_{mn}}$.

In precisely the same way, by giving y'_{rs} instead of x'_{rs} a variation in (23) to (26), we arrive at

$$\frac{dx_{mn}}{dy'_{rs}}=\text{co. }\xi'^m\eta'^n\text{ in expansion of}$$

$$-\frac{1}{r+1}\{y_{10}\xi'+y_{01}\eta'+y_{20}\xi'^2+\dots\}^s\frac{d}{d\eta'}\{x_{10}\xi'+x_{01}\eta'+x_{20}\xi'^2+\dots\}^{r+1}$$
$$\dots\dots\dots\dots\dots(36),$$

and $\frac{dy_{mn}}{dy'_{rs}} = \text{co. } \xi'^m \eta'^n$ in expansion of

$$-\frac{1}{s+1}\{x_{10}\xi'+x_{01}\eta'+x_{20}\xi'^2+\ldots\}^r \frac{d}{d\eta'}\{y_{10}\xi'+y_{01}\eta'+y_{20}\xi'^2+\ldots\}^{s+1} \qquad \ldots\ldots\ldots\ldots\ldots(37),$$

and at the general conclusion, symbolically written as in (35), that

$$\frac{d}{dy'_{rs}} = -\,\xi'^r\eta'^s\left\{\frac{d\xi}{d\eta'}+\lambda\frac{d\eta}{d\eta'}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(38).$$

To obtain, then, the transformed linear operator equivalent to

$$\Sigma P'_{rs}\frac{d}{dx'_{rs}} + \Sigma Q'_{rs}\frac{d}{dy'_{rs}},$$

it is effectual to put $\xi'^r\eta'^s$ both for $\frac{d}{dx'_{rs}}$ and $\frac{d}{dy'_{rs}}$ for all values of r and s involved, to then multiply the first sum by $-\frac{d\xi}{d\xi'}-\lambda\frac{d\eta}{d\xi'}$, and the second by $-\frac{d\xi}{d\eta'}-\lambda\frac{d\eta}{d\eta'}$, to expand the result in terms of ξ' and η' by (25) and (26), and finally in the expansion to put, for every $\lambda\xi'^m\eta'^n$, $\frac{d}{dy_{mn}}$, and for every $\xi'^m\eta'^n$ without a λ, $\frac{d}{dx_{mn}}$.

7. With a view to the utilization of these formulæ now arrived at, it is important to notice that, by first notions of partial differentiation,

$$\left.\begin{aligned}
\frac{d\xi}{d\xi'}\cdot\frac{d\xi'}{d\xi}+\frac{d\xi}{d\eta'}\cdot\frac{d\eta'}{d\xi} &= 1\\
\frac{d\xi}{d\xi'}\cdot\frac{d\xi'}{d\eta}+\frac{d\xi}{d\eta'}\cdot\frac{d\eta'}{d\eta} &= 0\\
\frac{d\eta}{d\xi'}\cdot\frac{d\xi'}{d\xi}+\frac{d\eta}{d\eta'}\cdot\frac{d\eta'}{d\xi} &= 0\\
\frac{d\eta}{d\xi'}\cdot\frac{d\xi'}{d\eta}+\frac{d\eta}{d\eta'}\cdot\frac{d\eta'}{d\eta} &= 1
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39);$$

for, the derivatives ... x_{rs}, y_{rs}, x'_{rs}, y'_{rs}, ... being all regarded as constant, (23) and (24) or (25) and (26) are merely two equations connecting ξ and η with ξ' and η'.

The same relations may also be written

$$\frac{\dfrac{d\xi}{d\xi'}}{\dfrac{d\eta'}{d\eta}}=-\frac{\dfrac{d\xi}{d\eta'}}{\dfrac{d\xi'}{d\eta}}=-\frac{\dfrac{d\eta}{d\xi'}}{\dfrac{d\eta'}{d\xi}}=\frac{\dfrac{d\eta}{d\eta'}}{\dfrac{d\xi'}{d\xi}}$$

$$=\frac{\dfrac{d\xi}{d\xi'}\cdot\dfrac{d\eta}{d\eta'}-\dfrac{d\xi}{d\eta'}\cdot\dfrac{d\eta}{d\xi'}}{1}=\frac{1}{\dfrac{d\xi'}{d\xi}\cdot\dfrac{d\eta'}{d\eta}-\dfrac{d\xi'}{d\eta}\cdot\dfrac{d\eta'}{d\xi}}\;\dots\dots\dots\;(40).$$

8. Let us now first apply what precedes to the reversion of operators included in the form

$$\Sigma\,(a+br+cs)\,x'_{rs}\frac{d}{dx'_{rs}}+\Sigma\,(a'+b'r+c's)\,y'_{rs}\frac{d}{dy'_{rs}}\;\dots\dots\dots(41),$$

where the summations extend over all positive integral (and zero) values of r and s for which $r+s>0$.

The symbolical form of the reversed operator is, by § 6,

$$-\left\{a\xi'+b\xi\frac{d\xi'}{d\xi}+c\eta\frac{d\xi'}{d\eta}\right\}\left(\frac{d\xi}{d\xi'}+\lambda\frac{d\eta}{d\xi'}\right)$$

$$-\left\{a'\eta'+b'\xi\frac{d\eta'}{d\xi}+c'\eta\frac{d\eta'}{d\eta}\right\}\left(\frac{d\xi}{d\eta'}+\lambda\frac{d\eta}{d\eta'}\right);$$

which may be written

$$-a\left\{\xi'\frac{d\xi}{d\xi'}+\lambda\xi'\frac{d\eta}{d\xi'}\right\}-a'\left\{\eta'\frac{d\xi}{d\eta'}+\lambda\eta'\frac{d\eta}{d\eta'}\right\}$$

$$-\frac{b+b'}{2}\left\{\frac{d\xi'}{d\xi}\,\frac{d\xi}{d\xi'}+\frac{d\eta'}{d\xi}\,\frac{d\xi}{d\eta'}+\lambda\left(\frac{d\xi'}{d\xi}\,\frac{d\eta}{d\xi'}+\frac{d\eta'}{d\xi}\,\frac{d\eta}{d\eta'}\right)\right\}\xi$$

$$-\frac{c+c'}{2}\left\{\frac{d\xi'}{d\eta}\,\frac{d\xi}{d\xi'}+\frac{d\eta'}{d\eta}\,\frac{d\xi}{d\eta'}+\lambda\left(\frac{d\xi'}{d\eta}\,\frac{d\eta}{d\xi'}+\frac{d\eta'}{d\eta}\,\frac{d\eta}{d\eta'}\right)\right\}\eta$$

$$-\frac{b-b'}{2}\left\{\frac{d\xi'}{d\xi}\,\frac{d\xi}{d\xi'}-\frac{d\eta'}{d\xi}\,\frac{d\xi}{d\eta'}+\lambda\left(\frac{d\xi'}{d\xi}\,\frac{d\eta}{d\xi'}-\frac{d\eta'}{d\xi}\,\frac{d\eta}{d\eta'}\right)\right\}\xi$$

$$-\frac{c-c'}{2}\left\{\frac{d\xi'}{d\eta}\,\frac{d\xi}{d\xi'}-\frac{d\eta'}{d\eta}\,\frac{d\xi}{d\eta'}+\lambda\left(\frac{d\xi'}{d\eta}\,\frac{d\eta}{d\xi'}-\frac{d\eta'}{d\eta}\,\frac{d\eta}{d\eta'}\right)\right\}\eta,$$

and, by (39) or (40), is consequently equal to

$$-a\left\{\xi'\frac{d\xi}{d\xi'}+\lambda\xi'\frac{d\eta}{d\xi'}\right\}-a'\left\{\eta'\frac{d\xi}{d\eta'}+\lambda\eta'\frac{d\eta}{d\eta'}\right\}-\frac{b+b'}{2}\xi-\frac{c+c'}{2}\lambda\eta$$

$$-\frac{b-b'}{2}\left\{\xi\frac{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}+\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}+\lambda\xi\frac{2\frac{d\eta}{d\xi}\frac{d\eta}{d\eta'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}\right\}$$

$$+\frac{c-c'}{2}\left\{\eta\frac{2\frac{d\xi}{d\xi'}\frac{d\xi}{d\eta'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}+\lambda\eta\frac{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}+\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}\right\}\quad\ldots\ (42).$$

The operators which in this multiply $\frac{1}{2}(b-b')$ and $\frac{1}{2}(c-c')$ are o some complexity. The preceding terms, however, represent

$$-a\left\{\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}\right\}-a'\left\{\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-\tfrac{1}{2}(b+b')\,\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}-\tfrac{1}{2}(c+c')\,\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}.$$

Accordingly, putting $c=c'$ and $b=b'$, we obtain the formula of reversion

$$a\Sigma\,.\,x'_{rs}\frac{d}{dx'_{rs}}+a'\Sigma\,.\,y'_{rs}\frac{d}{dy'_{rs}}+b\Sigma\left\{rx'_{rs}\frac{d}{dx'_{rs}}+ry'_{rs}\frac{d}{dy'_{rs}}\right\}$$

$$+c\Sigma\left\{sx'_{rs}\frac{d}{dx'_{rs}}+sy'_{rs}\frac{d}{dy'_{rs}}\right\}$$

$$=-a\Sigma\left\{rx_{rs}\frac{d}{dx_{rs}}+ry_{rs}\frac{d}{dy_{rs}}\right\}-a'\Sigma\left\{sx_{rs}\frac{d}{dx_{rs}}+sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-b\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}-c\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}\ \ldots\ldots(43).$$

The operators, therefore, which multiply a, a', b, c on the two sides of this identity are equivalent in pairs.

Now, a function of either unaccented or accented derivatives can have two kinds of homogeneity—in suffixed x's and in suffixed y's. It can also have four kinds of isobarism—in first and second suffixes of x's, and in first and second suffixes of y's. Denote by $i(x)$, $i(y)$, $w_1(x)$, $w_2(x)$, $w_1(y)$, $w_2(y)$ the several degrees and weights of a function of the unaccented derivatives, and let $i(x')$, $i(y')$, $w_1(x')$, &c. have like meanings as to the expression in terms of accented derivatives. The facts now arrived at may, it will be seen, be stated

$$\left.\begin{aligned} i(x') &= -w_1(x)-w_1(y) \\ i(y') &= -w_2(x)-w_2(y) \\ w_1(x')+w_1(y') &= -i(x) \\ w_2(x')+w_2(y') &= -i(y) \end{aligned}\right\} \quad \text{.....................(44)},$$

the last two being the conjugates, and so affording verifications, of the first two.

For instance, the meaning of the first equality is that a function of the accented derivatives which is homogeneous in those of x has for its reversion a function of the unaccented derivatives, throughout which the sum of first suffixes is constant and equal to the negative of the given degree. Similarly for the others. By addition of the first two, we see that the reversion of a function which is homogeneous on the whole, is on the whole isobaric, and conversely.

9. Closely akin to the operators reversed in the last article are the two

$$\Sigma . x'_{rs}\frac{d}{dy'_{rs}}, \quad \Sigma . y'_{rs}\frac{d}{dx'_{rs}}.$$

The transformation of the former is

$$-\Sigma . x'_{rs}\xi^r\eta^s\left(\frac{d\xi}{d\eta'}+\lambda\frac{d\eta}{d\eta'}\right)$$

$$= -\xi'\left(\frac{d\xi}{d\eta'}+\lambda\frac{d\eta}{d\eta'}\right)$$

$$= -\Sigma . sx_{rs}\frac{d}{dx_{r+1,s-1}} - \Sigma . sy_{rs}\frac{d}{dy_{r+1,s-1}} \quad \text{..............(45)},$$

while that of $\Sigma . y'_{rs}\dfrac{d}{dx'_{rs}}$ is

$$-\eta'\left(\frac{d\xi}{d\xi'}+\lambda\frac{d\eta}{d\xi'}\right)$$

$$= -\Sigma . rx_{rs}\frac{d}{dx_{r-1,s+1}} - \Sigma . ry_{rs}\frac{d}{dy_{r-1,s+1}} \quad \text{..............(46)}.$$

10. A function of the second and higher without the first derivatives is called a *pure* function. An aim in the sequel will be to discover properties of a pure function P, of the unaccented derivatives, which is such that, for some value of the constant μ, the reversion of

$$(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P$$

is a pure function of the accented derivatives.

Now, by (2),

$$x_{10}y_{01}-x_{01}y_{10}=\frac{1}{x'_{10}y'_{01}-x'_{01}y'_{10}}=\frac{x_{10}}{y'_{01}}=-\frac{x_{01}}{x'_{01}}=-\frac{y_{10}}{y'_{10}}=\frac{y_{01}}{x'_{10}}.$$

Thus, $$\frac{d}{dx'_{10}}\left\{x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\right\}=0$$

may be written $$\frac{d}{dx'_{10}}\left\{\left(\frac{x_{10}}{y'_{01}}\right)^{-\mu}P\right\}=0,$$

i.e., $$\frac{d}{dx'_{10}}(x_{10}^{-\mu}P)=0,$$

i.e., $$\frac{dP}{dx'_{10}}+\mu x_{10}P=0 \qquad \text{.........................(47)};$$

since, as will be seen in the next article,

$$\frac{dx_{10}}{dx'_{10}}=-x_{10}^2.$$

Again, $$\frac{d}{dx'_{01}}\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\right\}=0$$

means $$\frac{d}{dx'_{01}}(x_{10}^{-\mu}P)=0,$$

i.e., $$\frac{dP}{dx'_{01}}+\mu y_{10}P=0 \qquad \text{.........................(48)};$$

since, as will be seen, $$\frac{dx_{10}}{dx'_{01}}=-x_{10}y_{10}.$$

Once more, $$\frac{d}{dy'_{10}}\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\right\}=0$$

is the same as $$\frac{d}{dy'_{10}}(x_{10}^{-\mu}P)=0,$$

i.e., $$\frac{dP}{dy'_{10}}+\mu x_{01}P=0 \qquad \text{.........................(49)};$$

for it will be seen that

$$\frac{dx_{10}}{dy'_{10}} = -x_{10}x_{01}.$$

And lastly, $$\frac{d}{dy'_{01}}\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\right\} = 0$$

is $$\frac{d}{dy'_{01}}\left\{\left(\frac{x_{01}}{x'_{01}}\right)^{-\mu}P\right\} = 0,$$

i.e., $$\frac{d}{dy'_{01}}(x_{01}^{-\mu}P) = 0,$$

i.e., $$\frac{dP}{dy'_{01}} + \mu y_{01}P = 0 \quad\text{........................}(50);$$

since we shall see that

$$\frac{dx_{01}}{dy'_{01}} = -x_{01}y_{01}.$$

11. For settlement of the points here assumed, as well as for future use, it is desirable to examine more closely, by separation of the first and higher derivatives, the reversed operators which are found, as in §6, as the equivalents of $\frac{d}{dx'_{10}}$, $\frac{d}{dx'_{01}}$, $\frac{d}{dy'_{10}}$, $\frac{d}{dy'_{01}}$.

As the reversed equivalent of $\frac{d}{dx'_{10}}$, we have, by (35),

$$-\xi\left\{\frac{d\xi}{d\xi'}+\lambda\frac{d\eta}{d\xi'}\right\},$$

i.e., $$-\left\{x_{10}\xi'+x_{01}\eta'+(\xi-x_{10}\xi'-x_{01}\eta')\right\}$$
$$\times\left\{x_{10}+\lambda y_{10}+\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\},$$

i.e., $$-x_{10}^2\xi'-x_{10}x_{01}\eta'-x_{10}y_{10}\lambda\xi'-x_{01}y_{10}\lambda\eta'$$
$$-x_{10}(\xi-x_{10}\xi'-x_{01}\eta')-y_{10}\lambda(\xi-x_{10}\xi'-x_{01}\eta')$$
$$-x_{10}\left\{\xi'\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\xi'\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\}$$
$$-x_{01}\left\{\eta'\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\eta'\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\}$$
$$-\tfrac{1}{2}\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')^2-\lambda(\xi-x_{10}\xi'-x_{01}\eta')\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta');$$

that is to say, we have

$$\begin{aligned}\frac{d}{dx'_{10}} = &-x_{10}^2\frac{d}{dx_{10}} - x_{10}x_{01}\frac{d}{dx_{01}} - x_{10}y_{10}\frac{d}{dy_{10}} - x_{01}y_{10}\frac{d}{dy_{01}} \\ &-x_{10}\sum_{r+s\nless 2} x_{rs}\frac{d}{dx_{rs}} - y_{10}\sum_{r+s\nless 2} x_{rs}\frac{d}{dy_{rs}} \\ &-x_{10}\sum_{r+s\nless 2}\left\{rx_{rs}\frac{d}{dx_{rs}} + ry_{rs}\frac{d}{dy_{rs}}\right\} \\ &-x_{01}\sum_{r+s\nless 2}\left\{rx_{rs}\frac{d}{dx_{r-1,s+1}} + ry_{rs}\frac{d}{dy_{r-1,s+1}}\right\} \\ &-\Sigma\Sigma\left\{rx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}} + y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \quad \text{..............(51),}\end{aligned}$$

the limits of the last summations with regard to m and n and r and s being such as to allow the inclusion of all cases for which

$$r \nless 0 \text{ and } \ngtr m, \quad s \nless 0 \text{ and } \ngtr n,$$

$$r+s \nless 2 \text{ and } \ngtr m+n-2, \quad m \nless 1, \quad n \nless 0.$$

In like manner, the equivalent of $\frac{d}{dx'_{01}}$ is

$$-\eta\left\{\frac{d\xi}{d\xi'} + \lambda\frac{d\eta}{d\xi'}\right\},$$

i.e., $$-\{y_{10}\xi' + y_{01}\eta' + (\eta - y_{10}\xi' - y_{01}\eta')\}$$
$$\times\left\{x_{10} + \lambda y_{10} + \frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta')\right\},$$

i.e., $$\begin{aligned}&-x_{10}y_{10}\xi' - x_{10}y_{01}\eta' - y_{10}^2\lambda\xi' - y_{10}y_{01}\lambda\eta' \\ &-x_{10}(\eta - y_{10}\xi' - y_{01}\eta') - y_{10}\lambda(\eta - y_{10}\xi' - y_{01}\eta') \\ &-y_{10}\left\{\xi'\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta')\right\} \\ &-y_{01}\left\{\eta'\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta')\right\} \\ &-(\eta - y_{10}\xi' - y_{01}\eta')\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') - \tfrac{1}{2}\lambda\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta')^2;\end{aligned}$$

so that

$$\frac{d}{dx'_{01}} = -x_{10}y_{10}\frac{d}{dx_{10}} - x_{10}y_{01}\frac{d}{dx_{01}} - y_{10}^2\frac{d}{dy_{10}} - y_{10}y_{01}\frac{d}{dy_{01}}$$

$$-x_{10}\sum_{r+s\nless 2} y_{rs}\frac{d}{dx_{rs}} - y_{10}\sum_{r+s\nless 2} y_{rs}\frac{d}{dy_{rs}}$$

$$-y_{10}\sum_{r+s\nless 2}\left\{rx_{rs}\frac{d}{dx_{rs}} + ry_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-y_{01}\sum_{r+s\nless 2}\left\{rx_{rs}\frac{d}{dx_{r-1,s+1}} + ry_{rs}\frac{d}{dy_{r-1,s+1}}\right\}$$

$$-\Sigma\Sigma\left\{ry_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}} + y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \quad \ldots\ldots\ldots\ldots (52),$$

the limits of the last summations being as in (51).

Once more, as the equivalent of $\dfrac{d}{dy'_{10}}$ we have

$$-\xi\left\{\frac{d\xi}{d\eta'} + \lambda\frac{d\eta}{d\eta'}\right\},$$

i.e.,

$$-\left\{x_{10}\xi' + x_{01}\eta' + (\xi - x_{10}\xi' - x_{01}\eta')\right\}$$

$$\times\left\{x_{01} + \lambda y_{01} + \frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\},$$

i.e.,

$$-x_{10}x_{01}\xi' - x_{01}^2\eta' - x_{10}y_{01}\lambda\xi' - x_{01}y_{01}\lambda\eta'$$

$$-x_{01}(\xi - x_{10}\xi' - x_{01}\eta') - y_{01}\lambda(\xi - x_{10}\xi' - x_{01}\eta')$$

$$-x_{10}\left\{\xi'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-x_{01}\left\{\eta'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta')^2 - \lambda(\xi - x_{10}\xi' - x_{01}\eta')(\eta - y_{10}\xi' - y_{01}\eta');$$

so that

$$\frac{d}{dy'_{10}} = -x_{10}x_{01}\frac{d}{dx_{10}} - x_{01}^2\frac{d}{dx_{01}} - x_{10}y_{01}\frac{d}{dy_{10}} - x_{01}y_{01}\frac{d}{dy_{01}}$$

$$-x_{01}\sum_{r+s\nless 2} x_{rs}\frac{d}{dx_{rs}} - y_{01}\sum_{r+s\nless 2} x_{rs}\frac{d}{dy_{rs}}$$

$$-x_{10}\sum_{r+s\nless 2}\left\{sx_{rs}\frac{d}{dx_{r+1,s-1}} + sy_{rs}\frac{d}{dy_{r+1,s-1}}\right\}$$

$$-x_{01}\sum_{r+s\nless 2}\left\{sx_{rs}\frac{d}{dx_{rs}} + sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-\Sigma\Sigma\left\{sx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}} + y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \quad \text{..................(53),}$$

the limits of the last summations being as in (51), except for

$$m \nless 0, \quad n \nless 1,$$

instead of $\quad m \nless 1, \quad n \nless 0.$

And lastly, since $\dfrac{d}{dy'_{01}}$ reverses into

$$-\eta\left\{\frac{d\xi}{d\eta'} + \lambda\frac{d\eta}{d\eta'}\right\},$$

i.e.,

$$-\{y_{10}\xi' + y_{01}\eta' + (\eta - y_{10}\xi' - y_{01}\eta')\}$$

$$\times\left\{x_{01} + \lambda y_{01} + \frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\},$$

i.e.,

$$-x_{01}y_{10}\xi' - x_{01}y_{01}\eta' - y_{10}y_{01}\lambda\xi' - y_{01}^2\lambda\eta'$$

$$-x_{01}(\eta - y_{10}\xi' - y_{01}\eta') - y_{01}\lambda(\eta - y_{10}\xi' - y_{01}\eta')$$

$$-y_{10}\left\{\xi'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-y_{01}\left\{\eta'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-(\eta - y_{10}\xi' - y_{01}\eta')\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') - \tfrac{1}{2}\lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')^2,$$

we have

$$\begin{aligned}\frac{d}{dy'_{01}} = & -x_{01}y_{10}\frac{d}{dx_{10}} - x_{01}y_{01}\frac{d}{dx_{01}} - y_{10}y_{01}\frac{d}{dy_{10}} - y_{01}^2\frac{d}{dy_{01}} \\ & -x_{01}\sum_{r+s \not< 2} y_{rs}\frac{d}{dx_{rs}} - y_{01}\sum_{r+s \not< 2} y_{rs}\frac{d}{dy_{rs}} \\ & -y_{10}\sum_{r+s \not< 2}\left\{sx_{rs}\frac{d}{dx_{r+1,s-1}} + sy_{rs}\frac{d}{dy_{r+1,s-1}}\right\} \\ & -y_{01}\sum_{r+s \not< 2}\left\{sx_{rs}\frac{d}{dx_{rs}} + sy_{rs}\frac{d}{dy_{rs}}\right\} \\ & -\Sigma\Sigma\left\{sy_{m-r,n-s}\left[x_{rs}\frac{d}{dy_{m,n-1}} + y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \text{.........(54),}\end{aligned}$$

with limits of the last summations, as in (53).

12. As a first application of this last article, notice that, as partially used in § 10,

$$\left.\begin{array}{llll} \dfrac{d}{dx'_{10}}x_{10} = -x_{10}^2, & \dfrac{d}{dx'_{10}}x_{01} = -x_{10}x_{01}, & \dfrac{d}{dx'_{10}}y_{10} = -x_{10}y_{10}, & \dfrac{d}{dx'_{10}}y_{01} = -x_{01}y_{10} \\ \dfrac{d}{dx'_{01}}x_{10} = -x_{10}y_{10}, & \dfrac{d}{dx'_{01}}x_{01} = -x_{10}y_{01}, & \dfrac{d}{dx'_{01}}y_{10} = -y_{10}^2, & \dfrac{d}{dx'_{01}}y_{01} = -y_{10}y_{01} \\ \dfrac{d}{dy'_{10}}x_{10} = -x_{10}y_{01}, & \dfrac{d}{dy'_{10}}x_{01} = -x_{01}^2, & \dfrac{d}{dy'_{10}}y_{10} = -x_{10}y_{01}, & \dfrac{d}{dy'_{10}}y_{01} = -x_{01}y_{01} \\ \dfrac{d}{dy'_{01}}x_{10} = -x_{01}y_{10}, & \dfrac{d}{dy'_{01}}x_{01} = -x_{01}y_{01}, & \dfrac{d}{dy'_{10}}y_{10} = -y_{10}y_{01}, & \dfrac{d}{dy'_{01}}y_{01} = -y_{01}^2 \end{array}\right\} \text{..................(55).}$$

13. We can now exhibit in their full cogency the conditions of § 10, that the reversion of

$$(x_{10}y_{01} - x_{01}y_{10})^{-\mu}P$$

be independent of all the first derivatives

$$x'_{10},\ x'_{01},\ y'_{10},\ y'_{01}.$$

For, upon inserting in (47) to (50) from (51) to 54), we get vanishing expressions in which the terms free from first derivatives and the terms multiplying

$$x_{10},\ x_{01},\ y_{10},\ y_{01}$$

must separately vanish, since no term of any one set can go out against a term of any other. The conclusions are found to be that the pure function P must be annihilated separately by each of the twelve operators

$$\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}-\mu \ldots\ldots\ldots\ldots\ldots (56),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}+\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}-\mu \ldots\ldots\ldots\ldots\ldots (57),$$

$$\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}-\mu \ldots\ldots\ldots\ldots\ldots (58),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}+\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}-\mu \ldots\ldots\ldots\ldots\ldots (59),$$

$$\Sigma\,.\,x_{rs}\frac{d}{dy_{rs}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dx_{rs}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61),$$

$$\Sigma\,.\,rx_{rs}\frac{d}{dx_{r-1,s+1}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{r-1,s+1}} \ldots\ldots\ldots\ldots\ldots (62),$$

$$\Sigma\,.\,sx_{rs}\frac{d}{dx_{r+1,s-1}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{r+1,s-1}} \ldots\ldots\ldots\ldots\ldots (63),$$

$$\Sigma\Sigma\left\{rx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}}+y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \ldots\ldots\ldots\ldots (64),$$

$$\Sigma\Sigma\left\{ry_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}}+y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \ldots\ldots\ldots\ldots (65),$$

$$\Sigma\Sigma\left\{sx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}}+y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \ldots\ldots\ldots\ldots (66),$$

$$\Sigma\Sigma\left\{sy_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}}+y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \ldots\ldots\ldots\ldots (67).$$

The extent of every summation may be concisely expressed by saying, that both in the coefficients and the symbols of partial differentiation none but second and higher derivatives occur, and that no possible term, with such a limitation, is absent.

Of the twelve annihilators it is not at once clear how many are independent. The first four are equivalent to three only, since

$$(56)-(57)-(58)+(59)$$

is identically zero.

14. Most of these annihilators admit of ready interpretation.

The first four (three independent) express facts with regard to the degrees and weights of P; namely, that throughout P the degree in derivatives of x is equal to that in derivatives of y, that the sum of first suffixes (of x's and y's together) is equal to the sum of second suffixes, and that the sum of either of the equal degree sums and either of the equal weight sums is constant and equal to μ. These may be expressed

$$i(x) = i(y) = \mu - w_1(x) - w_1(y) = \mu - w_2(x) - w_2(y) \; \ldots \; (68).$$

That these equal characteristics are all constant will be established in the following article.

Again, (60) and (61) tell us that P is a function of the determinants included in the system

$$\left\| \begin{matrix} x_{20}, & x_{11}, & x_{02}, & x_{30}, & x_{21}, & x_{12}, & x_{03}, & \ldots \\ y_{20}, & y_{11}, & y_{02}, & y_{30}, & y_{21}, & y_{12}, & y_{03}, & \ldots \end{matrix} \right\| \; \ldots\ldots\ldots\ldots\ldots \; (69);$$

in other words, that P is unaltered, but for a power of the modulus by any linear transformation of the cogredient pairs of quantities x_{20}, y_{20}; x_{11}, y_{11}; x_{02}, y_{02}; ...; that is to say, by such a substitution as

$$\left. \begin{aligned} x+px'+qy'+r &= lX+mY \\ y+p'x'+q'y'+r' &= l'X+m'Y \end{aligned} \right\} \; \ldots\ldots\ldots\ldots\ldots\ldots \; (70).$$

Once more, (62) and (63) require that P be a full invariant of the two systems of quantics

$$\left. \begin{matrix} (x_{20}, x_{11}, x_{02}\big)(\xi', \eta')^2 \\ (x_{30}, x_{21}, x_{12}, x_{03}\big)(\xi', \eta')^3 \\ \&\text{c.} \qquad \&\text{c.} \end{matrix} \right\} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (71),$$

$$\left. \begin{matrix} (y_{20}, y_{11}, y_{02}\big)(\xi', \eta')^2 \\ (y_{30}, y_{21}, y_{12}, y_{03}\big)(\xi', \eta')^3 \\ \&\text{c.} \qquad \&\text{c.} \end{matrix} \right\} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (72).$$

Lastly, (64) to (67) are annihilators closely related to the annihilators

$$V \equiv \Sigma\Sigma r z_{rs} z_{n-r,m-s} \frac{d}{dz_{n-1,m}} \qquad\qquad (73),$$

$$V_2 \equiv \Sigma\Sigma s z_{rs} z_{n-r,m-s} \frac{d}{dz_{n,m-1}} \qquad\qquad (74),$$

of pure cyclicants or ternary reciprocants. A detailed study of this connexion would undoubtedly be productive of valuable results.

15. It will now be established that the pure functions P and P' which satisfy

$$(x_{10}y_{01}-x_{01}y_{10})^{-\mu} P = P' \qquad\qquad (75),$$

which may also be written

$$(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}\mu} P = (x'_{10}y'_{01}-x'_{01}y'_{10})^{-\frac{1}{2}\mu} P' \qquad\qquad (76),$$

or, again,

$$P = (x'_{10}y'_{01}-x'_{01}y'_{10})^{-\mu} P' \qquad\qquad (77),$$

must be necessarily the same functions of the unaccented and accented derivatives respectively; in other words, that such a P is a pure reciprocant in two independent and two dependent variables.

P must be homogeneous, and isobaric both in first and in second suffixes, by the fact that it is an invariant of the system of quantics (71) and (72). Moreover, by a particular case of the invariant property for transformations (70), P is unaffected, except perhaps as to sign, by the interchange of x and y. Thus the relations (68) may be replaced by the far more stringent system

$$2w_1(x) = 2w_1(y) = 2w_2(x) = 2w_2(y)$$
$$= \mu - i(x) = \mu - i(y) = \text{constant} \qquad\qquad (78).$$

For exactly the same reasons, regarding (77) instead of its equivalent (75), the like facts hold as to the characteristics of P': viz.,

$$2w_1(x') = 2w_1(y') = 2w_2(x') = 2w_2(y')$$
$$= \mu - i(x') = \mu - i(y') = \text{constant} \qquad\qquad (79).$$

The constant degrees and weights of P' must, moreover, be the same as those of P. For, with regard to P,

$$i(x) = \mu - w_1(x) - w_1(y)$$

$$= \mu + \text{degree of } (x'_{10}y'_{01} - x'_{01}y'_{10})^{-\mu} P' \text{ in suffixed } x\text{'s},$$

by (44) and (77),

$$= \mu + i(x') - \mu,$$

where $i(x')$ refers to P',

$$= i(x') \quad \text{...} (80).$$

Thus, the equal constant characteristics in (78) and those in (79) are equal to the same constant.

That P and P' are of the same form may now be seen, as follows. We observe that the equalities (5) to (10) may be expressed by saying that, for the values 2, 0; 1, 1; 0, 2 of r and s,

$$r!\,s!\,x'_{rs} = (x_{10}y_{01} - x_{01}y_{10})^{-r-s-1}\left(m'\frac{d}{dx'} - m\frac{d}{dy'}\right)^r \left(-l'\frac{d}{dx'} + l\frac{d}{dy'}\right)^s$$

$$\times (l'y - m'x) \quad \text{..............................} (81),$$

$$r!\,s!\,y'_{rs} = (x_{10}y_{01} - x_{01}y_{10})^{-r-s-1}\left(m'\frac{d}{dx'} - m\frac{d}{dy'}\right)^r \left(-l'\frac{d}{dx'} + l\frac{d}{dy'}\right)^s$$

$$\times (-ly + mx) \quad \text{..........................} (82);$$

in which, after the operations are performed, l, l', m, m' are given the values x_{10}, x_{01}, y_{10}, y_{01}.

It is to be observed that the second—*i.e.*, $(r+s)^{\text{th}}$—derivatives of x and y are involved linearly in these expressions, and that there is no term free from them.

Now, referring to (3), $x'_{r+1,s}$, $y'_{r+1,s}$ are derived from x'_{rs}, y'_{rs} by the operator

$$\frac{1}{r+1}(x_{10},\ y_{01} - x_{01}y_{10})^{-1}\left(y_{01}\frac{d}{dx'} - y_{10}\frac{d}{dy'}\right),$$

and $x'_{r,s+1}$, $y'_{r,s+1}$ by the operator

$$\frac{1}{s+1}(x_{10}y_{01} - x_{01}y_{10})^{-1}\left(-x_{01}\frac{d}{dx'} + x_{10}\frac{d}{dy'}\right).$$

By repetition of these processes of derivation we obtain, in the notation of (81) and (82), that for higher values than 2 of the sum $r+s$ the values of $r!\,s!\,x'_{rs}$ and $r!\,s!\,y'_{rs}$ consist of terms given by the expressions on the dexter of (81) and (82), which are accordingly linear in the $(r+s)^{\text{th}}$ derivatives and involve besides only first derivatives, with the addition of terms which are of higher degree than the first in second and higher derivatives.

Now, $(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P$ being a function equal to P' and so homogeneous, and of degree i say, in the values of $x'_{20}, y'_{20}, \dots x'_{rs}, y'_{rs}, \dots$, which is also homogeneous and of the same degree i in $x_{20}, y_{20}, \dots x_{rs}, y_{rs}, \dots$, it follows that the additional terms not written in (81) and (82), as these could only give rise to terms of degree higher than i, must cancel. P will therefore be, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, the same function of

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s(l'y-m'x) \;\dots\dots\; (83),$$

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s(-ly+mx) \dots\dots (84),$$

for different values of r and s, as P' is of x'_{rs}, y'_{rs}, and the other corresponding accented derivatives.

But, whatever z be,

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s z$$

$$=\frac{1}{r!\,s!}\Big\{(-1)^s(r+s)!\,m'^r l'^s z_{r+s,0}$$

$$+(-1)^{s+1}(r+s-1)!\,1!\left(l\frac{d}{dl'}+m\frac{d}{dm'}\right)(m'^r l'^s)\,z_{r+s-1,1}$$

$$+(-1)^{s+2}(r+s-2)!\,2!\,\frac{1}{2!}\left(l\frac{d}{dl'}+m\frac{d}{dm'}\right)^2(m'^r l'^s)\,z_{r+s-2,2}+\dots\Big\}$$

$= \text{co. } \xi^r\eta^s$ in

$$z_{r+s,0}\,(m'\xi'-l'\eta')^{r+s}+z_{r+s-1,1}\,(m'\xi'-l'\eta')^{r+s-1}(-m\xi'+l\eta')$$

$$+z_{r+s-2,2}\,(m'\xi'-l'\eta')^{r+s-2}(-m\xi'+l\eta')^2+\dots \quad \dots(85).$$

Accordingly, restoring to l, l', m, m' their values, we see that P is, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, the same function of the various coefficients in the set of pairs of quantics

$$\left.\begin{array}{c}
(\ x_{01}y_{20}-y_{01}x_{20},\quad x_{01}y_{11}-y_{01}x_{11},\quad x_{01}y_{02}-y_{01}x_{02}) \\
\times\,(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^2 \\
(-x_{10}y_{20}+y_{10}x_{20},\ -x_{10}y_{11}+y_{10}x_{11},\ -x_{10}y_{02}+y_{10}x_{02}) \\
\times\,(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^2 \\
(\ x_{01}y_{30}-y_{01}x_{30},\quad x_{01}y_{21}-y_{01}x_{21},\quad x_{01}y_{12}-y_{01}x_{12},\quad x_{01}y_{03}-y_{01}x_{03}) \\
\times\,(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^3 \\
(-x_{10}y_{30}+y_{10}x_{30},\ -x_{10}y_{21}+y_{10}x_{21},\ -x_{10}y_{12}+y_{10}x_{12},\ -x_{10}y_{03}+y_{10}x_{03}) \\
\times\,(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^3
\end{array}\right\} \\
\text{\&c.,}\qquad\qquad \text{\&c.} \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots(86),$$

as P' is of those of the set of pairs of quantics

$$\left.\begin{array}{c}
(x'_{20},\ x'_{11},\ x'_{02})\,(u,\ v)^2 \\
(y'_{20},\ y'_{11},\ y'_{02})\,(u,\ v)^2 \\
(x'_{30},\ x'_{21},\ x'_{12},\ x'_{03})\,(u,\ v)^3 \\
(y'_{30},\ y'_{21},\ y'_{12},\ y'_{03})\,(u,\ v)^3
\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (87). \\
\text{\&c.,}\qquad\qquad \text{\&c.}$$

We know, however, by the invariant property expressed by (62) and (63), and by that expressed by (60) and (61) (*cf.* § 14), that P is the same function, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, of the coefficients of the pairs of quantics

$$\left.\begin{array}{c}
(x_{20},\ x_{11},\ x_{02})\,(\xi',\ \eta')^2 \\
(y_{20},\ y_{11},\ y_{02})\,(\xi',\ \eta')^2 \\
(x_{30},\ x_{21},\ x_{12},\ x_{03})\,(\xi',\ \eta')^3 \\
(y_{30},\ y_{21},\ y_{12},\ y_{03})\,(\xi',\ \eta')^3
\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (88), \\
\text{\&c.,}\qquad\qquad \text{\&c.}$$

as it is of the pairs of quantics (86).

Hence, finally, in (75), (76), or (77), P and P' are the same functions of the unaccented and accented second and higher derivatives respectively.

16. Any discussion of the multiplicity of pure reciprocants P, whose existence is here conditioned, must find place in a possible future communication. The investigation will naturally be on the lines, and by aid of the results, of my papers on pure cyclicants. I here note only the first specimen,

$$(x_{20}y_{11}-x_{11}y_{20})(x_{11}y_{02}-x_{02}y_{11})-(x_{20}y_{02}-x_{02}y_{20})^2 \quad \ldots\ldots\ldots(90),$$

the eliminant of the first pair of quantics in (88). It may also be written

$$4\,(x_{20}x_{02}-\tfrac{1}{4}x_{11}^2)(y_{20}y_{02}-\tfrac{1}{4}y_{11}^2)-(x_{20}y_{02}+x_{02}y_{20}-\tfrac{1}{2}x_{11}y_{11})^2 \quad \ldots\ldots(91),$$

in which form its connexion with the first pure cyclicant $x_{20}x_{02}-\frac{1}{4}x_{11}^2$ is exhibited.

On Newton's Classification of Cubic Curves.
By Mr. W. W. Rouse Ball.

[*Read December* 11*th*, 1890.]

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I. *Subject-matter of Newton's Tract.*

1. The *Enumeratio linearum tertii ordinis*, first published by Newton in 1704 as an appendix to his *Opticks*, contains the earliest attempt to apply the methods of analytical geometry to an exhaustive classification of cubic curves in a manner analogous to that in which conics are divided into three species. I propose to give an outline of Newton's argument so far as it is concerned with cubics, to compare it with his earlier treatment of the subject as indicated by manuscripts in the Portsmouth collection, and to conclude with a few notes on the history and bibliography of the subject.

2. The essay on cubic curves is the first appendix to the *Opticks*; the second appendix being the tract *De Quadratura Curvarum*. In the advertisement, which is prefixed to the book, Newton explains the reason of his appending to a work on optics propositions on the quadrature of curves and on the subject of cubics by saying: "Some Years ago I lent out a Manuscript containing such Theorems, and having since met with some Things copied out of it, I have on this Occasion made it publick. . . . And I have joined with it another small Tract concerning the Curvilinear Figures of the Second Kind, which was also written many Years ago, and made known to some Friends, who have solicited the making it publick."

It is not known at what time the earliest of his manuscripts on cubic curves was written, but there is some reason for thinking that it was before 1676. It is probable that the manuscript from which the edition was printed was written in or after 1695.

3. The essay as published is a catalogue of results, and it is possible that it was issued rather with the view of establishing the priority of Newton's investigations than as an exposition of the subject. No proofs of the propositions are given, and such explanatory remarks as are occasionally made with a view of showing how the results may be demonstrated are very concise. In fact, the tract occupies only twenty-four pages.

4. Newton divides cubics into fourteen genera. These genera comprise seventy-two species, to which we ought to add six additional ones which are left out from his enumeration. In the tract, a diagram of a curve of every species is carefully drawn, and these diagrams (except for the six species omitted) present to the eye the figures of all the possible forms of cubic curves, when degenerate forms are not taken into account.

The method of classification is uniform, simple, and intelligible to any one acquainted with the elements of analytical geometry and of the theory of equations. These are great merits. Moreover, the arrangement into genera is not only easy, but leads at once to the characteristic properties of the curves. Thus, apart from its historical interest as being the earliest systematic discussion of a group of curves of a degree higher than the second, the essay is well worthy of careful study.

5. The tract is divided into seven sections. In the first, which is on the orders of lines, Newton explains that the orders of lines are measured by the degrees of their respective equations: but (since he does not reckon a straight line as a curve) he describes conics either as lines of the second order, or as curves of the first genus. Similarly, cubics are described either as lines of the third order or as curves of the second genus.

In the second section he treats of properties of cubics analogous to those of conics.

In the third section he shows that the equation of a cubic always can be written in one of four canonical forms: namely,

$$xy^2+ey = ax^3+bx^2+cx+d \quad\text{......................(i.),}$$

$$xy = ax^3+bx^2+cx+d \quad\text{......................(ii.),}$$

$$y^2 = ax^3+bx^2+cx+d \quad\text{......................(iii.),}$$

or

$$y = ax^3+bx^2+cx+d \quad\text{......................(iv.).}$$

These equations are hereafter alluded to as forms (i.), (ii.), (iii.), and (iv.). In this section he also describes his nomenclature. He designates the various classes of curves discussed as hyperbolas and parabolas, and distinguishes them from the conic sections by prefixing descriptive adjectives. This terminology, though it still remains in use, is neither clear nor convenient.

In the fourth section he discusses the nature of the curves indicated by these several equations, and he divides the curves into species by considering the maximum and minimum values of x.

In the fifth section he enunciates the remarkable proposition that, just as the shadow of a circle cast by a luminous point gives rise to all the conics, so the shadow cast by a luminous point of the curves represented by the equation $y^2 = ax^3+bx^2+cx+d$ gives rise to all the cubics. He also refers to double points, both in the plane and at infinity.

In the sixth section he treats of the description of curves, especially of cubics satisfying given conditions.

In the seventh and last section he deals with the graphical solution of problems by the use of curves.

6. The subject of this paper is an account of the earlier investigations of Newton on cubics, as contained in his unpublished manuscripts and memoranda; a description of the methods used in the third, fourth, and fifth sections of the tract; the results arrived at; and the subsequent history of the subject, so far as it is concerned with Newton's method.

II. *Reduction of Cubic Curves to One of Four Canonical Forms.*

7. The first problem in the classification of cubic curves is the reduction of their equations to four canonical forms. To effect this, Newton originally (in the first rough draft of the essay, which is described later) changed the origin and axes, and he has left two independent proofs of this method. In the tract as published, he solved this problem by a process which depended on finding the diameter conjugate to an asymptote.

8. The first of these methods is by transformation of coordinates. Before describing the details of the procedure adopted by Newton, I may, for convenience of future reference, indicate briefly the necessary steps. The general equation of a cubic curve in cartesian coordinates (which for simplicity we will suppose to be rectangular) is

$$u_3+u_2+u_1+u_0=0.$$

The cubic equation $u_3=0$ determines the directions in which the curve is cut by the line at infinity. At least one root of this cubic equation is real. If there is only one real root, the line determined by that root is taken as the axis of y; similarly, if there are three real roots which are not all equal, the line determined by that root which is not equal to the other two is taken as the axis of y. In either of these cases x will be a factor of u_3, but x^2 will not be a factor of u_3; hence u_3 will take the form $x(Ax^2+Bxy+Cy^2)$, where $C\neq 0$. In the case where the three roots of $u_3=0$ are all equal, the direction given by the roots is taken as the axis of y, and u_3 will then take the form Ax^3. Thus the equation of the curve can be always written in the form

$$z(Ax^2+Bxy+Cy^2)+Ex^2+Fxy+Gy^2+Hx+Ky+L=0,$$

where either $C \neq 0$. or $B=0$ and $C=0$.

First, suppose $C \neq 0$; then, if the origin is changed to a point (p, q), we can choose p and q to make the coefficient of xy (namely, $2Bp+2Cq+F$), and the coefficient of y^2 (namely, $Cp+G$) vanish; next, if the axis of x is turned through an angle $\arctan(-B/2C)$, and the axis of y is kept unaltered, the coefficient of x^2y will vanish, and the equation takes the form (i.).

Next, suppose $B=0$ and $C=0$; then we have three cases:—(α) If $G \neq 0$, we can, by a transformation similar to that described in the last paragraph, make the coefficients of y and of xy vanish, and the equation takes the form (iii.). (β) But, if $G=0$ and $F \neq 0$, we can, by changing the origin to $(-K/F, 0)$, express the equation in the form (ii.). (γ) Lastly, if $G=0$ and $F=0$, the equation takes the form (iv.).

9. Newton's original method is substantially the same as that given above; but the algebraic analysis involved is more complicated, since, instead of first taking the axis of y parallel to an asymptote, then changing the origin, and finally turning the axis of x through some convenient angle, he takes the general equation of the third degree, and changes the origin and the direction of the axis in one step.

I had intended at one time to print his manuscript as it stands, but, on reading it again carefully, I think that the following summary of his method brings out sufficiently the points of interest in it.

10. After a brief description of the meaning of coordinates, and their use in determining curves by means of equations, he says that the most general equation of a cubic curve referred to rectangular axes is

$$av^3+bzv^2+cz^2v+dz^3+ev^2+fzv+gz^2+hv+kz+l=0,$$

(z, v) being the coordinates of a point on the curve. He then changes the origin to a point (r, s), and takes as axes of ξ and η lines making angles α and β with the axis of z. He thus obtains two relations equivalent to the following:—

$$z=r+\xi\cos\alpha+\eta\cos\beta, \quad v=s+\xi\sin\alpha+\eta\sin\beta \;\;\ldots\ldots\ldots(a).$$

[As a matter of fact, he does not use the trigonometrical ratios, but constructs triangles having one side equal to the unit of length, and having one angle equal to α or β, as the case may be. It is only to save circumlocution that I here use the trigonometrical functions.] But, instead of employing these values of z and v, he puts

$$\xi \cos \alpha = x \quad \text{and} \quad \eta \sin \beta = y,$$

and thus obtains

$$z = r + x + y \cot \beta = \text{(say) } r + x + qy,$$

and $$v = s + x \tan \alpha + y = \text{(say) } s + px + y.$$

This is, he says, more convenient, because he can give p and q any values that he likes, while, if he used the formulæ (α), he would have to consider whether α and β were so chosen as to make each of the ratios $\sin \alpha$, $\sin \beta$, $\cos \alpha$, and $\cos \beta$ numerically less than unity. Moreover, he remarks, that, if he knows the relation between x and y, he can at any time deduce the corresponding relation between ξ and η, which is the equation of the curve; and the form of the equation will not be altered by the change.

Next, he substitutes these values of z and v in the given equation and obtains an equation in x and y which has no less than 84 terms. He takes for q one of the roots of

$$a + bq + cq^2 + dq^3 = 0,$$

and then in terms of this root he determines values of p, r, and s (and the expressions for them are neither short nor simple) which make certain other coefficients vanish. He thus obtains the four canonical forms.

11. The algebraic analysis above described involves considerable labour, but the details of the work are not given in the manuscript. Newton's second proof by transformation of axes is much more ingenious. The analysis is printed (I believe for the first time) in Appendix I., Arts. 89–95, and is very simple. The following is a summary of it:—

Suppose that the given equation of a cubic (referred to $A\xi$ and $A\eta$ as coordinate axes) is

$$a\eta^3 + b\xi\eta^2 + c\xi^2\eta + d\xi^3 + e\eta^2 + f\xi\eta + g\xi^2 + h\eta + k\xi + l = 0,$$

where $$AB = \xi \quad \text{and} \quad BC = \eta.$$

On AB take a point D, so that

$$DB = p\eta,$$

where p is a root of the equation

$$a+bp+cp^2+dp^3 = 0.$$

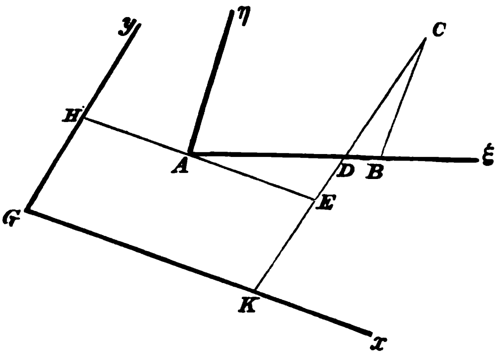

FIG. 1.

Therefore the triangle BCD is determined; hence BC is a known multiple of DC, say

$$BC = q\,.\,DC.$$

Let $AD = x'$ and $DC = y'$.

Therefore $\eta = qy'$ and $\xi = x'+p\eta = x'+pqy'$.

Substitute these values of ξ and η in the equation of the cubic, and we obtain an equation (in x' and y' as current coordinates) of the form

$$b'x'y'^2+c'x'^2y'+d'x'^3+e'y'^2+f'x'y'+g'x'^2+h'y'+k'x'+l' = 0.$$

(a) If $b' \neq 0$, then (see Fig. 1) on CD produced take a point E, so that

$$DE : AD = c' : 2b'.$$

Therefore the triangle AED is determined; hence AD or x' is a known multiple of AE, say

$$AD = n\,.\,AE.$$

Let $AE = x''$ and $EC = y''$.

Therefore $x' = nx''$ and $y' = y'' - DE = y'' - \frac{c'}{2b'}nx''$.

Substitute these values of x' and y' in the equation at the end of the last paragraph, and we obtain an equation (in x'' and y'' as current coordinates) of the form

$$b''x''y''^2 + d''x''^3 + e''y''^2 + f''x''y'' + g''x''^2 + h''y'' + k''x'' + l'' = 0.$$

Next, change the origin to $\left(-\frac{e''}{b''}, -\frac{f''}{2b''}\right)$, that is, on EA produced take a point H, so that

$$AH : 1 = e'' : b'',$$

and on CE produced take a point K, so that

$$EK : 1 = f'' : 2b'';$$

and complete the parallelogram $EHGK$. Hence, if

$$GK = x \text{ and } KC = y,$$

we have $x'' = x - \frac{e''}{b''}$, and $y'' = y - \frac{f''}{2b''}$.

Substitute these values of x'' and y'', and the equation reduces to the form numbered (i.) in Art. 5.

(β) If $b' = 0$, then, if $c' \neq 0$, take (with a notation analogous to the above)

$$DE : AD = d' : c', \quad AH : 1 = f'' : 2c'', \quad EK : 1 = g'' : c'',$$

and the equation, when referred to GH and GK as axes of x and y, reduces to the form (i.).

(γ) If $b' = 0$ and $c' = 0$, then, if $e' \neq 0$, take

$$DE : AD = f' : 2e', \quad AH : 1 = g'' : 3d'', \quad EK : 1 = h'' : 2e'',$$

and the equation, when referred to GK and GH as axes of x and y, reduces to the form (iii.).

If, however, $b' = 0$, $c' = 0$, and $e' = 0$, we must slightly alter the above construction. On DA produced take a point H (Fig. 2), defined as mentioned below. On CD produced take points E and K, defined as mentioned below. Complete the parallelogram $EHGK$, and refer the equation to GK and GH as axes of x and y.

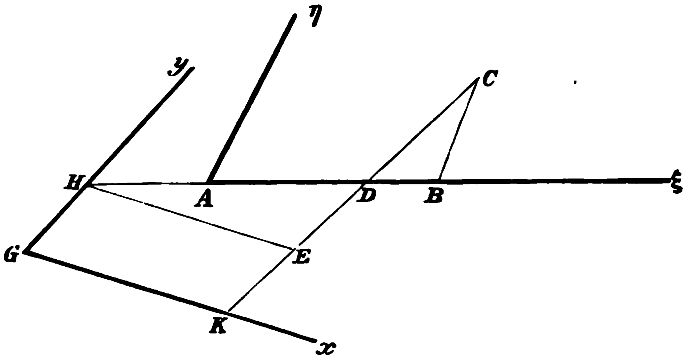

FIG. 2.

(δ) If $b' = 0$, $c' = 0$, $e' = 0$, then, if $f' \neq 0$, take

$$AH : 1 = h' : f', \quad DE : HD = g'' : f'', \quad EK : 1 = k'' : f'',$$

and the equation reduces to the form (ii.).

(ε) Lastly, if $b' = 0$, $c' = 0$, $e' = 0$, and $f' = 0$, take

$$AH : 1 = g' : 3d', \quad DE : HD = k'' : h'', \quad EK : 1 = l'' : h'',$$

and the equation reduces to the form (iv.).

In the cases (γ), (δ), and (ε) the equation is expressed finally in a somewhat simpler form than is necessary to show merely that it is included in one of the four canonical forms given in Art. 5.

12. Another method of reduction is given in Stirling's *Lineae Tertii Ordinis Neutonianae* (Prop. xv., p. 83). There is some reason for thinking that Stirling consulted Newton before publishing this work, and therefore I refer to it here in passing.

13. In the later manuscript copies of the tract, and as ultimately published, the reduction to the canonical forms was effected by a quasi-geometrical proof. Briefly sketched out, it is as follows. Draw a series of chords parallel to the real asymptote. If each chord cuts the curve in two points, the locus of their middle points will be a

hyperbola, and if the asymptotes of this hyperbola are taken as axes of coordinates, the equation takes the form (i.). But if each chord only meets the curve in one point, then, taking the asymptote as the axis of y, the equation takes the form (ii.); in fact, the asymptote is a tangent at a point of inflexion at infinity. Next, if the cubic is touched by the line at infinity (*si crura illa opposita parabolici sint generis*), then, if any chord meets the parabolic branch in two points, the locus of the point midway between them will be a straight line, and if this line is taken as the axis of x and a line parallel to the chord as the axis of y, the equation takes the form (iii.). But if the chord meets the curve in only one point, the equation similarly takes the form (iv.).

14. This may be clearer if it is expressed in analytical language. If a line parallel to the real asymptote is taken as the axis of y, in the manner explained in Art. 8, the equation takes the form

$$x\,(Ax^2+Bxy+Cy^2)+Ex^2+Fxy+Gy^2+Hx+Ky+L=0,$$

where either $C \neq 0$, or $B=0$ and $C=0$.

This equation may be written

$$y^2\,(Cx+G)+y\,(Bx^2+Fx+K)+Ax^3+Ex^2+Hx+L=0.$$

The locus of the middle points of chords parallel to the asymptote is

$$2y\,(Cx+G)+Bx^2+Fx+K=0;$$

and if the axes of this hyperbola are taken as coordinate axes, the equation of the cubic takes the form (i.). This covers the case when $C \neq 0$. If, however, $C=0$, then also $B=0$, and the locus of the middle points of chords parallel to the asymptote is a straight line: namely,

$$2Gy+Fx+K=0.$$

If $G \neq 0$, this line is taken as the axis of x; this is case (iii.). If $G=0$ and $F \neq 0$, this line is taken as the axis of y; this is case (ii.). Lastly, if $G=0$ and $F=0$, this locus is the line at infinity; and the equation is in the form (iv.).

III. *Classification of Cubics.*

15. Having thus, in Section III. of the tract, proved that every cubic equation is reducible to one of four canonical forms, Newton proceeds in Section IV. to classify the curves represented by these four equations into genera and species—degenerate forms being left out of account.

In this classification, the division into classes and genera (as well as that into the four canonical forms) depends on the nature of the infinite branches.

16. The separation into species of the curves which constitute any genus or subdivision is effected by solving the equation of the curve for y, and thus determining the maximum and minimum values of x.

The equations numbered (ii.), (iii.), (iv.) in Art. 5 give y as an explicit function of x. In the equation numbered (i.) this is also true if $e = 0$; and, if $e \neq 0$, Newton discusses the equation in the form

$$(xy+\tfrac{1}{2}e)^2 = ax^4+bx^3+cx^2+dx+\tfrac{1}{4}e^2.$$

Thus, in all the groups hereafter described, the criterion which determines the species of the curves constituting any genus will be dependent on the nature and position of the roots of one of the two following equations: namely,

$$ax^4+bx^3+cx^2+dx+\tfrac{1}{4}e^2 = 0, \quad \text{where } e \neq 0;$$

and

$$ax^3+bx^2+cx+d = 0.$$

17. The four canonical forms or cases are divided into seven (or six) classes. These classes are subdivided into fourteen (or thirteen) genera, which contain the seventy-eight species, from which however the degenerate forms of a conic and a straight line and of three straight lines are excluded. It will be convenient to present here, in a tabular form, the final result of the classification; it should be added that in his text Newton describes every division (whether a class, genus, or species) by the word "species."

Canonical Form or Case.	*Class.*	*Genus.*	*Species.*
I. $xy^2+ey = ax^3+bx^2+cx+d$	Redundant Hyperbolas (a positive)	adiametral	9
		monodiametral	12 + 2
		tridiametral	2 + 2
		[asymptotes concurrent]	9
	Defective Hyperbolas (a negative)	adiametral	6
		monodiametral	7
	Parabolic Hyperbolas ($a = 0$)	adiametral	7
		monodiametral	4 + 2
	Hyberbolisms of Conics ($a = 0$, $b = 0$)	hyperbolic	4
		elliptic	3
		parabolic	2
II. $xy = ax^3+bx^2+cx+d.$	(The Trident)		1
III. $y^2 = ax^3+bx^2+cx+d.$	Diverging Parabolas		5
IV. $y = ax^3+bx^2+cx+d.$	(The Cubic Parabola)		1

18. I have given this table in the form which I understand to be indicated by Newton's language, but I should remark (i.) that the redundant and the defective hyperbolas would constitute more properly a single class of central cubic hyperbolas; and (ii.) that the hyperbolic and elliptic genera of the hyperbolisms would constitute more properly a single genus of hyperbolisms of central conics.

Case I.

19. The first case comprises all curves whose equations can be expressed in the form

$$xy^2+ey = ax^3+bx^2+cx+d.$$

These curves are divided into the following classes:—(i.) those in which a is positive, called *redundant hyperbolas*, or *hyperbolic hyperbolas*, or *hyperbolae triformes*; (ii.) those in which a is negative, called *defective hyperbolas* or *elliptic hyperbolas*; (iii.) those in which a is zero, which are further divided into (α) those in which b is not zero, called *parabolic hyperbolas*; and (β) those in which b is zero, called *hyperbolisms of conics.* It is usual to describe this last subdivision as a fourth class.

20. The curves in each class are further subdivided into genera according to the nature of their "absolute" diameters. If an asymptote cuts a cubic curve again in a point at a finite distance from the origin, the curve has no absolute diameter corresponding to that asymptote; but if an asymptote cuts the curve in three points at infinity (*i.e.*, is a tangent at a point of inflexion at infinity), the curve has an absolute diameter corresponding to that asymptote. Thus to say that a curve is adiametral signifies merely that it has no point of inflexion at infinity.

An adiametral curve is described by Newton as *non-bifidus* or *circumflexus*, a monodiametral curve as *uno modo bifidus* or *directus*, and a tridiametral curve as *trifariam bifidus.*

21. Case I. Class 1. *Redundant Hyperbolas.*—These are cubics whose equations can be expressed in the form

$$xy^2+ey = ax^3+bx^2+cx+d,$$

where a is positive.

These curves have three real and distinct asymptotes: namely,

$$x = 0 \quad \text{and} \quad y = \pm\sqrt{a}\,(x + b/2a).$$

These asymptotes cut the cubic again in points determined by their respective intersections with the lines

$$y = d/e \quad \text{and} \quad x = (4ad \mp 2be\sqrt{a})/(b^2 - 4ac \pm 4ae\sqrt{a}).$$

22. *First Subdivision. Adiametral redundant hyperbolas.*—Since a curve of this kind has no diameters, none of the three points on the curve at infinity is a point of inflexion there. Hence each asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0 \quad \text{and} \quad b^2 - 4ac \neq \mp 4ae\sqrt{a}.$$

These curves are divided into *nine species*, according as all, or some, or none of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2 = 0$ are real or equal. In the typical case when all the roots are real, unequal, and of the same sign, there is a closed oval lying inside the triangle formed by the asymptotes. Nicole (*Mémoires de l'Académie des Sciences* for 1729, Paris, 1731, p. 217, Art. 31) asserted that there was in addition a tenth species formed by the oval coalescing with the ambigenous hyperbolic branch: this is not the case, the oval can coalesce with the circumscribing hyperbolic branch, but not with either of the others.

23. *Second Subdivision. Monodiametral redundant hyperbolas.*—Since a curve of this kind has only one diameter, one and only one of the three points on the curve at infinity is a point of inflexion there. If the asymptote at this point is taken as the axis of y, it will not cut the curve in any point at a finite distance from the origin. Therefore

$$e = 0 \quad \text{and} \quad b^2 \neq 4ac.$$

These curves are divided into *fourteen species* dependent on the nature and position of the roots of the equation $ax^3 + bx^2 + cx + d = 0$. Of these fourteen species, Newton enumerated only twelve; the remaining two being added by Stirling in 1717 (*Lineae tertii ordinis Neutonianae*, p. 99, species 11). In the *Ladies' Diary* for 1788 (pp. 44–46), and for 1789 (pp. 44–45) there is an attempt to show that there are three more species, making the total number in the subdivision to be seventeen. The examples given show curves of a slightly different form to those drawn by Newton, but they are in no sense new species.

24. *Third Subdivision. Tridiametral redundant hyperbolas.*—Since a curve of this kind has three diameters, each of the three points on the curve at infinity is a point of inflexion there. [Newton was aware that, if two of them are points of inflexion, the third is so also.]

Hence $$e = 0 \text{ and } b^2 = 4ac.$$

These curves are divided into *four species*, dependent on the nature and position of the roots of the equation $x\,(2ax+b)^2+4ad = 0$. Of these four species, Newton enumerated only two; the other two being added by Stirling in 1717 (*Lineae tertii ordinis Neutonianae*, p. 102, species 24).

25. *Fourth Subdivision.*—Newton made a fourth subdivision, which comprises the particular cases of the preceding three subdivisions when the asymptotes are concurrent. He enumerated *nine species*; four of these being particular cases of the first subdivision, four of the second, and one of the third.

26. Case I. Class 2. *Defective Hyperbolas.*—These are cubics whose equations can be expressed in the form

$$xy^2+ey = ax^3+bx^2+cx+d,$$

where a is negative.

These curves have only one real asymptote: namely, $x = 0$. This asymptote cuts the cubic again in the point $(0, d/e)$. Hence the curve has no diameter if $e \neq 0$, and has one diameter if $e = 0$.

27. *First Subdivision. Adiametral defective hyperbolas.* — Since a curve of this kind has no diameter, the point on the curve at infinity is not a point of inflexion there. Hence the asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0.$$

These curves are divided into *six species*, dependent on the nature and position of the roots of the equation $ax^4+bx^3+cx^2+dx+\frac{1}{4}e^2 = 0$.

28. *Second Subdivision. Monodiametral defective hyperbolas.*—Since a curve of this kind has one diameter, the point on the curve at infinity is a point of inflexion there. Hence the asymptote cannot

cut the curve again at a finite distance from the origin. Therefore

$$e = 0.$$

These curves are divided into *seven species*, dependent on the nature and position of the roots of the equation $ax^3+bx^2+cx+d = 0$.

29. Case I. Class 3. *Parabolic Hyperbolas.*—These are cubics whose equations can be expressed in the form

$$xy^2+ey = bx^3+cx+d,$$

where $b \neq 0$.

These curves have one linear or proper asymptote and one parabolic asymptote. The proper asymptote is $x = 0$, and it cuts the cubic again in the point $(0, d/e)$. The parabolic asymptote is $y^2 = bx+c$, and it degenerates into two parallel straight lines if $b = 0$.

30. *First Subdivision. Adiametral parabolic hyperbolas.*—Since a curve of this kind has no diameter, the point on the curve at infinity is not a point of inflexion there. Hence the asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0.$$

These curves are divided into *seven species*, dependent on the nature and position of the roots of the equation $bx^3+cx^2+dx+\frac{1}{4}e^2 = 0$.

31. *Second Subdivision. Monodiametral hyperbolas.*—Since a curve of this kind has one diameter, the point on the curve at infinity is a point of inflexion there. Hence the asymptote does not cut the curve again at a finite distance from the origin. Therefore

$$e = 0.$$

These curves are divided into *six species*, dependent on the nature and position of the roots of the equation $bx^2+cx+d = 0$. Of these six species, Newton enumerated only four. One of these omitted curves consists of two infinite branches and an oval, the other of two infinite branches and an acnode. Of these, the one with an oval was mentioned by Nicole in 1731 (*Mémoires de l'Académie des Sciences*, for 1731, Paris, 1733, p. 503, Fig. 10), and by Murdoch in 1740 (*Newtoni Genesis Curvarum per Umbras*, p. 77); the one with an acnode was mentioned by Nicholas Bernoulli (see Murdoch, p. 87). Both species were described fully by Edmund Stone in 1736 (*Philosophical Transactions*, 1840, vol. XLI., pp. 318–320), and by De Gua De Malves in 1740 (*Usages de l'analyse de Descartes*, pp. 367, 368). It is probable that these discoveries of the two species not mentioned by Newton were made independently.

32. CASE I. CLASS 4. *Hyperbolisms of Conics.*[*]—These are cubics whose equations can be expressed in the form

$$xy^2+ey = cx+d.$$

They might be regarded as a subdivision of parabolic hyperbolas, being those parabolic hyperbolas whose asymptotic parabola degenerates into two parallel straight lines (namely, $y = \pm\sqrt{c}$); but Newton treated them as constituting a class by themselves.

They have three asymptotes: namely,

$$x = 0, \quad y = \sqrt{c}, \quad \text{and} \quad y = -\sqrt{c},$$

and they have also a double point at infinity.

Newton divided them into three subdivisions, according as this double point is a crunode, an acnode, or a spinode.

33. *First Subdivision. Hyperbolisms of hyperbolas.*—In this case

$$c \text{ is positive,}$$

the double point at infinity is a crunode, and the three asymptotes are real. These curves are divided into *four species*, according as (i.) $e \neq 0$, $d^2 > ce^2$; (ii.) $e \neq 0$, $d \neq 0$, $d^2 < ce^2$; (iii.) $e \neq 0$, $d = 0$; (iv.) $e = 0$.

34. *Second Subdivision. Hyperbolisms of ellipses.*—In this case

$$c \text{ is negative,}$$

the double point at infinity is an acnode, and only one asymptote is real. These curves are divided into *three species*, according as (i.) $e \neq 0$, $d \neq 0$; (ii.) $e \neq 0$, $d = 0$; (iii.) $e = 0$, $d \neq 0$. The last of these is the "Witch of Agnesi."

35. *Third Subdivision. Hyperbolisms of parabolas.*—In this case

$$c = 0,$$

the double point at infinity is a spinode, and the three asymptotes are real, though two of them are coincident. These curves are divided into *two species*, according as $e = 0$ or $e \neq 0$.

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[*] If $y = \phi(x)$ is the equation of a curve, then $xy = \phi(x)$ is a *hyperbolism* of that curve.

Thus, since $y^2+ey = cx^2+dx$ is any conic, therefore $(xy)^2+e(xy) = cx^2+dx$ is a hyperbolism of a conic. In the text, the factor x is rejected.

Case II.

36. The second case comprises all curves whose equations can be expressed in the form

$$xy = ax^3 + bx^2 + cx + d.$$

There is only *one species*, usually known as the *Trident*, but sometimes called the *Cartesian Parabola* (in memory of Descartes), and sometimes a *Parabolism of a hyperbola.*

Case III.

37. The third case comprises all curves whose equations can be expressed in the form

$$y^2 = ax^3 + bx^2 + cx + d.$$

These curves are called *Diverging Cubic Parabolas*, or sometimes *Neilian Parabolas* (in memory of W. Neil, the earliest mathematician who succeeded in rectifying a curve). They are divided into *five species*, dependent on the nature and position of the roots of the equation $ax^3 + bx^2 + cx + d = 0$.

Case IV.

38. The fourth case comprises all curves whose equations can be expressed in the form

$$y = ax^3 + bx^2 + cx + d.$$

There is only *one species*, usually known as the *Cubic Parabola*, but sometimes called the *Wallisian Parabola* (in memory of John Wallis, the Savilian Professor at Oxford).

IV. *Discrimination of Genera.*

39. In the earliest manuscript copy Newton proceeds next to give criteria by which it is possible to tell to which genus or subdivision a curve belongs, without reducing its equation to one of the canonical forms. I give Newton's discussion at length in Appendix II., Arts. 101–105, 107. It does not appear in any of the printed editions, and I am not aware that it has been published previously.

40. Stirling, in his *Lineae Tertii Ordinis Neutonianae* (pp. 126–128), mentions this problem; but seems to think that the discrimination should be effected by a reduction to the canonical form. This [illegible]duction appears to be the only way of discriminating between the [illegible]cies.

41. The rule that Newton gives is as follows. If the coefficient of y^3 is not zero, the equation can be written in the form

$$y^3+bxy^2+cx^2y+dx^3+ey^2+fxy+gx^2+hy+kx+l=0,$$

that is, $x^3(z^3+bz^2+cz+d)+x^2(ez^2+fz+g)+x(hz+k)+l=0,$

where z stands for y/x. We shall denote this by

$$ux^3+vx^2+wx+l=0.$$

Let suffixes indicate differentiation to z, and let

$$s=u_2v^2-2u_1vv_1+2u_1^2w.$$

Then u, v, w, their differential coefficients, and s are functions of z.

(α) If the three roots of $u=0$ are real and unequal, the curve will be a redundant hyperbola. The hyperbola will be adiametral, monodiametral, or tridiametral, according as the equation $s=0$ is satisfied by none of these roots, by only one of them, or by all of them.

(β) If only one root of $u=0$ is real, then (i.), if that root does not satisfy the equation $u_2^2=8u_1$, the curve will be a defective hyperbola; but (ii.), if it does satisfy the equation $u_2^2=8u_1$, the curve will be a hyperbolism of an ellipse; in both these cases the curve being adiametral or monodiametral according as this root does not or does satisfy the equation $s=0$.

(γ) If $u=0$ has three real roots and two of them are equal, then (i.), if the root which is not equal to the other two roots does not satisfy the equation $v=0$, the curve will be a parabolic hyperbola; but (ii.), if it does satisfy the equation $v=0$, but not the equation $v_1^2=2u_2w$, the curve will be a hyperbolism of a hyperbola; and (iii.), if it satisfies both $v=0$ and $v_1^2=2u_2w$, the curve will be a hyperbolism of a parabola; in all these cases the curve being adiametral or monodiametral according as this value of z does not or does satisfy the equation $s=0$.

(δ) Lastly, if $u=0$ has three real roots and all of them are equal, then (i.), if this root does not satisfy the equation $v=0$, the curve will be a diverging parabola; but (ii.), if it does satisfy $v=0$ but does not satisfy $v_1=0$, the curve will be a trident; while (iii.), if it satisfies both $v=0$ and $v_1=0$, the curve will be a cubic parabola.

Should the coefficient of y^3 in the original equation be zero, but not that of x^3, then the equation must be arranged in powers of x, we must put $z = x/y$, and corresponding tests can be applied.

If the coefficients of x^3 and of y^3 are both zero, then, if $b \neq 0$ and $c \neq 0$, the curve will be a redundant hyperbola; this is similar to case (α). But if either $b = 0$ or $c = 0$, then the curve will be a parabolic hyperbola or a hyperbolism of a hyperbola or of a parabola; this is similar to case (γ). The further determination of the genus may be effected in the way given above.

42. In this discussion Newton shows also how the values of u_1, u_2, v, v_1 and w, corresponding to a root of $u = 0$, enable us to give a geometrical construction for the corresponding asymptote (see Appendix II., Art. 106).

43. I may add here that Newton uses the binomial coefficients and not the differential coefficients, so that his formulæ are slightly different from those that I have given in Art. 41. Thus, if $u = z^n$, he uses $\frac{1}{2}n(n-1)z^{n-2}$ where I have used u_2. This is an instance of John Bernoulli's "regulam falsam" (see Montucla's *Histoire des Mathématiques*, vol. III., p. 105, or H. Sloman's *Claim of Leibnitz to the Invention of the Differential Calculus*, Cambridge, 1860, p. 121).

V. *Generation of Cubics by Shadows.*

44. After having enumerated in Section IV. of the tract all the possible species of cubics, Newton begins Section V. by noting that every cubic curve may be regarded as the shadow of one of the five diverging parabolas cast by a luminous point on a plane properly situated.

His enunciation of this remarkable property is as follows:—If the shadows of curves caused by a luminous point are projected on an infinite plane, the shadows of conic sections will always be conic sections; those of curves of the second genus will always be curves of the second genus; those of curves of the third genus will always be curves of the third genus; and so on. And in the same manner as the circle, by projecting its shadow, generates all the conic sections, so the five divergent parabolas, by their shadows, generate all other curves of the second genus. And thus some of the more simple curves of other genera might be found which would form all curves of the same genus by the projection of their shadows on a plane.

45. No proof of this proposition is given in the printed essay, but among the Portsmouth collection of Newton's papers is a sheet of double foolscap containing rough holograph notes, enumerating the various species which arise from the section by a plane in some defined position of one of the five cones having these diverging parabolas for bases.

Thus, under the head of the second species of the diverging parabolas, he writes, "Si oculus infinite distet vel plana projectionis et projicientis parallela sint, projectio erit Parabola ejusdem speciei cum projiciente. . . . Si secat in nodo, projectio erit Hyperbola triformis bifida duas ex asymptotis parallelas habens"—*i.e.*, the species which Newton terms number 64, and which is described above as the fourth species of the Hyperbolisms of a Hyperbola.

But no demonstrations are given, and the numerous erasures, corrections, and interlineations make the manuscript somewhat difficult to read.

46. I have little doubt that Newton had arrived at this remarkable result, which proved a puzzle to most of his contemporaries, by the method of projection indicated in the *Principia*, Bk. I., sect. 5, lemma XXII. (See also propositions XXV., XXVI., XXVII.)

47. It is not difficult to prove the property by analytical geometry (see, for example, Salmon's *Higher Plane Curves*, second edition, Arts. 195, 196); but to Newton's contemporaries, who relied largely on pure geometry, it was by no means easy to establish its truth, and, in fact, a quarter of a century elapsed before any one published a demonstration of it.

48. The earliest writer who gave a proof of it was Nicole, who communicated it to the French Academy on 1 Dec., 1731. (*Mémoires de l'Académie des Sciences*, for 1731, Paris, 1733, pp. 494–510.)

49. A few days later, on 12 Dec. 1731, Clairaut communicated another demonstration, in which the result is deduced from certain general theorems. This is contained in his paper entitled *Sur les courbes que l'on forme en coupant une surface courbe quelconque par un plan donné de position*: see the *Mémoires de l'Académie des Sciences* for 1731, Paris, 1733, pp. 490–493.

50. In 1740 another demonstration of this result was given by

P. Murdoch in his *Neutoni Genesis Curvarum per Umbras*, published at Leyden. This work was republished at London in 1746.

51. Lastly, in 1755, still another demonstration was given by Jacquier in the appendix to his *Elementi di perspettiva.*

52. This property enables us to divide cubics into five groups such that all the curves in each group have their fundamental characteristics (*i.e.*, those characteristics which are unaltered by projection) in common. This provides a better basis for classification than the one used by Newton. Chasles has made the interesting note that only five cubics have centres, and that these five central cubics can also by their shadows generate every form of a cubic. In fact, if the five cones described on the diverging parabolas as bases are cut by a certain plane, we obtain the five canonical central cubics. (See *Aperçu historique sur l'origine et le développement des méthodes en géometrie*, Brussels, 1837, note xx., p. 348.)

53. In more recent times this property has formed the subject of elaborate memoirs by Möbius in 1852 (*Die Grundformen der Linien der dritten Ordnung*, Collected works, vol. II., pp. 89–176), and by Cayley in 1864 (*On Cubic Cones and Curves*, *Cambridge Philosophical Transactions*, vol. XI., pp. 129–144).

VI. *Bibliography.*

54. I shall conclude this paper by enumerating the various original manuscripts and the editions of Newton's tract with which I am acquainted.

55. I have already mentioned that three holograph manuscripts of the tract are preserved at Cambridge in the Portsmouth collection (Section I., Class IV., Numbers 1, 2).* There is also a fragment of a fourth manuscript (I., IV., 4), and some notes on the generation by shadows (I., IV., 3). The notes on the generation by shadows and the subsequent history of that theorem have been already mentioned. I shall describe the other manuscripts in what, I have no doubt, is their chronological order. Indeed the successive increases in the number

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* The two earliest manuscripts are catalogued in a single entry as "an early *copy*."

of species enumerated would serve alone to distinguish the order of production.

56. The earliest of these manuscripts is entitled *Enumeratio Curvarum Trium Dimensionum.* This is a rough draft of the tract as afterwards published, but it contains the substance of only the third and fourth sections. Moreover, the subdivision into species is not discussed, and the final result is nine divisions and sixteen genera, arranged and named as follows, the nomenclature being somewhat different to that subsequently used:—

1. Hyberbola triformis	sine diametro. cum diametro. cum tribus diametris.
2. Cissoidalis vel Endea	circumflexa. directa.
3. Parabola Cissoideos	circumflexa. directa.
4. Hyperbola Hyperbolae	sine diametro. cum diametro.
5. Conchoidalis vel Hyperbola Ellipseos	circumflexa. directa.
6. Hyperbola Conchae vel Parabolae	sine diametro. cum diametro.
7. Parabola reflexa vel recurvata.	
8. Parabola fissa vel Cartesiana.	
9. Parabola circumflexa sive Simplex cubica.	

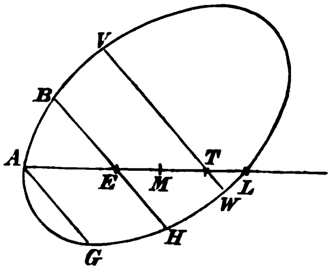
57. The language of the text is not exactly the same as that subsequently published, but the only material differences which I have noticed between this and the copy printed in the *Opticks* are (i.) the method by which the equation is reduced to the four canonical forms (see above, Art. 10); (ii.) a slightly fuller treatment of adiametral redundant hyperbolas, such as giving the coordinates of the points where the asymptote cuts the curve again; and (iii.) the addition of criteria for discriminating the genera (see above, Art. 41).

58. The next manuscript is that which is described in the Portsmouth catalogue as *Fragments concerning lines of the third order.*

This is intermediate in point of time between the first and second manuscripts. These fragments consist of two or three closely written foolscap sheets. Newton commences by proving some general properties of diameters and segments of chords, with special reference to conics and cubics (Conicarum sectionum communes proprietates mutatis mutandis ad Curvas magis compositas applicari possunt). These are a rough draft of what afterwards appeared as the first part of the second manuscript, and are the foundation of the results given in Section II. of the published tract. At the end of these he has written: "Hae sunt insigniores proprietates Conicarum sectionum, et videtis quod curvis etiam superioris ordinis conveniunt." These words are crossed out.

59. Next, he proceeds to give the geometrical interpretations of the general equations of the second and third degrees. These are as follows.

60. Suppose that AG, BH are two parallel chords of a conic, and that VW is any chord drawn parallel to them. Through A draw another chord AL cutting BH in E, and VW in T. On AL take a point M so that



$$AM : EM = AG : EH - EB,$$

Let $TV = y$; then he asserts that y is determined by the equation

$$y^2 + \frac{AG}{AM} MT \,.\, y + \frac{EB \,.\, EH}{EA \,.\, EL} AT \,.\, TL = 0,$$

where the lines are regarded as directed magnitudes.

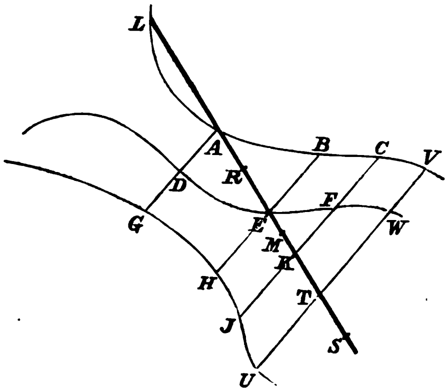
In fact, if AL and AG are taken as axes of x and y, the equation of any conic can be written in the form

$$y^2 - y(ax + b) + cx^2 + dx = 0.$$

In the above case,

$$AG = b, \quad AM = -\frac{b}{a}, \quad AL = -\frac{d}{c}, \quad \frac{EB \, . \, EH}{EA \, . \, EL} = -c.$$

61. Similarly, let ADG, BEH, CFJ be three parallel chords of a cubic, and VWU any chord drawn parallel to them. Join AE. Let the chord AE cut the cubic again in L, the chord CJ in K, and the chord VU in T. On AE take a point M so that



$$AM : EM = AD + AG : EH - EB.$$

Also, on AE take points R and S, so that

$$AR \, . \, AS : AD \, . \, AG = ER \, . \, ES : EB \, . \, EH$$

$$= KR \, . \, KS : KC \, . \, KF - KC \, . \, KJ - KF \, . \, KJ.$$

Let $TV = y$; then he asserts that y is determined by the equation

$$y^3 - \frac{AD + AG}{AM} TM \, . \, y^2 - \frac{AD \, . \, AG}{AR \, . \, AS} TR \, . \, TS \, . \, y$$

$$- \frac{KC \, . \, KF \, . \, KJ}{KA \, . \, KL \, . \, KE} TA \, . \, TL \, . \, TE = 0,$$

from which all the properties of every cubic may be deduced.

It is interesting to note that the geometrical properties ultimately published in the second section of the tract are deducible from this relation, and were probably thus obtained.

62. The further discussion of this relation would involve the use of solid geometry; and (apparently for this reason) he abandons this geometrical method of investigating the properties of cubics, and commences afresh by taking the general equation of the third degree. His discussion of the reduction of that equation to the four canonical forms has been already mentioned (see above, Art. 11). He concludes by briefly indicating the principles of classification to be adopted, and gives a tabular analysis of the result. This table shows the division into classes, genera, and species; but only sixty species are enumerated. The missing species are—(i.) nine redundant hyperbolas, namely, two adiametral, six monodiametral, and two tridiametral; (ii.) two defective hyperbolas, namely, one of each subdivision; and (iii.) one diverging parabola. The more interesting parts of these *Fragments* are printed in Appendix I., Arts. 84–99.

63. The second manuscript is entitled, *Enumeratio Curvarum secundi Ordinis*. The original title was *Enumeratio Curvarum tertii Ordinis*. This was first corrected by replacing *Curvarum* by *Linearum*, and then changed again to the form given above. The numerous verbal corrections interlined in this manuscript alter the text to a form which corresponds closely with that in which it was published afterwards.

The geometrical propositions collected at the beginning of the *Fragments* are placed at the commencement of this manuscript, and an introductory paragraph is added. The geometrical equations of a conic and a cubic (see above, Arts. 60, 61) are omitted, and the reduction of cubic equations to the four canonical forms is effected in the way given in the printed edition, and which I described above in Art. 13. The nomenclature is also somewhat simplified, and is substantially the same as that afterwards printed. The division into species is carried out in considerable detail, but only sixty-nine species are enumerated, the missing species being three monodiametral redundant hyperbolas, of which one is the limiting form of the other two when their asymptotes are concurrent. The two paragraphs about the generation of cubics by means of the shadows of the diverging parabolas, which form part of Section V. of the printed edition, and the last paragraph in Section VII., are missing; and as the manuscript is closely and continuously written, and ends with "Finis," we may assume that these paragraphs were not contained in any pages which are now lost. Sections V. and VI. are written out twice. The manuscript is divided into thirty articles with marginal headings. The diagrams (if ever there were any) are lost. The loose sheet in which the manuscript is wrapped does not belong to the manuscript. There still remain numerous minute verbal differences between this and the text of the printed editions, but I do not think any material differences other than those described above.

64. The third manuscript is entitled, *Enumeratio Linearum tertii Ordinis.* The title was originally written as *Enumeratio Curvarum cubici generis.* This was then altered to the heading given in the second manuscript. Finally, the three last words were erased, and the title changed to the form given above. The first section of the printed tract, the proposition about generation by shadows, and the last paragraph of Section VII., all of which are absent from the second manuscript, are inserted here. The curves are classified into seventy-two species, but the diagrams are absent. The whole is carefully written out, and, except for a few verbal alterations, the text is the same as that subsequently published. There is however no reference to a division into sections, but the whole is divided into thirty-four articles.

65. With this manuscript are two loose sheets, used as scribbling paper, which contain, among other notes, two draft tabular analyses of the classification of cubics. These do not differ materially from those adopted in the text, but they are drawn up in a form closely resembling that which I have given in Art. 17, with the addition of descriptive adjectives for most of the species.

66. There is nothing to show the dates of the first two manuscripts or of the *Fragments*, but one of these loose scribbling sheets of the third manuscript is written on the back of a short business letter, dated "London, 6 June, '95," and directed to "Isaac Newton, Esq., a.M., Trinnity Collidge, Cambridge." It is probable, therefore, that the whole of this manuscript is of a date subsequent to this, but I see no reason to suppose that it was written later than the summer or autumn of 1695.

67. The following printed editions of the essay on cubic curves are in the library of Trinity College, Cambridge, and include, so far as I know, all that have been published.

68. The enumeration of cubic curves was originally published, in

1704, as an appendix to the first edition of the *Opticks*, the title-page of which runs as follows:—*Opticks: or, a Treatise of the Reflexions, Refractions, Inflexions, and Colours of Light. Also Two Treatises of the Species and Magnitude of Curvilinear Figures.*

In the advertisement to the second edition, which is dated July 16, 1717, Newton says: "In this Second Edition of these Opticks I have omitted the Mathematical Tracts published at the End of the former Edition, as not belonging to the Subject." The tracts were left out in all the subsequent English editions, and I believe that they were not added to any of the Continental editions of the *Opticks*.

69. In 1706 a Latin translation of the *Opticks*, prepared by Samuel Clarke and revised by Demoivre, was published in London. The two tracts above mentioned are appended to this, but they were omitted from subsequent editions.

70. In 1711 the essay on Cubic Curves was reprinted in the collection of Newton's tracts, which was edited by William Jones, and issued at London, under the title, *Analysis per Quantitatum Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis.* The essay is reprinted without comment. Of all the editions this is the most pleasant to use, as the type, diagrams, and get-up are excellent. In 1723 this book was reprinted at Amsterdam, and was issued both by itself and also bound up with a reprint of the second edition of the *Principia*, which was published there in the same year.

71. In 1744 the essay was reprinted in the *Opuscula Mathematica* of Newton, edited by J. Castillioneus, and issued in three volumes at Lausanne.

72. In 1779 it was reprinted in vol. I. of the collected edition of Newton's works, issued by S. Horsley at London.

73. In 1861 a translation, with notes, was published by C. R. M. Talbot at London. The title-page of some of the earlier copies is dated 1860.

74. The chief commentary on the work is one by Stirling, which was published at Oxford in 1717, under the title, *Lineae Tertii Ordinis Neutonianae, sive Illustratio Tractatus D. Neutoni De Enumeratione Linearum Tertii Ordinis.* This contains demonstrations of most of the results which had been merely stated by Newton. A considerable part of the work is occupied by introductory matter, and the portion

devoted to the classification of cubics occupies less than forty pages. Stirling was an intimate friend of Newton, and there is reason to think that the proofs and comments were approved, and to some extent supplied, by Newton. This work was reprinted at Paris in 1797.

75. A commentary on part of the work, and especially on the algebraical reduction of a cubic to one of four canonical forms, was given by Nicole under the title *Traité des lignes du troisième ordre*, in the *Mémoires de l'Académie des Sciences*, for 1729, Paris, 1731, pp. 194–224.

76. Two objections have been urged against the system of classification adopted by Newton. In the first place, he nowhere defines what is meant by a species, and he is not consistent in his use of the term. In the second place, the curves in each group have not the same fundamental characteristics, that is, such as are unaltered by projection.

77. De Gua De Malves, in his *Usages de l'analyse de Descartes*, published at Paris in 1740, pointed out the first of these objections, and he proposed (pp. 365–367) to consider the maximum and minimum values of y as well as of x. This would increase largely the number of species, but would not materially improve the system of classification.

On the second of these objections, see above, Arts. 52, 53.

78. Cramer, in his *Introduction à l'analyse des lignes courbes algébriques*, published at Geneva in 1750, and Euler, in his *Analysis Infinitorum*, published at Lausanne in 1748, proposed to classify curves solely by reference to the character of their infinitely distant points. They both refused to recognise the various curves formed by the degeneration of an oval into an acnode, or its coalescence with other branches of the curve, as constituting distinct species. Cramer (p. 359–369) gives 14 species of cubics, and Euler (Bk. II., chap. ix., pp. 114–126) gives 16 species. This, after all, is not very different from the method used by Newton, except that Cramer and Euler would describe as species what I have termed genera.

79. In more recent times Plücker suggested a new system of classification of cubic curves, with special reference to critic centres. He divides cubics into 6 classes, 61 groups or genera, and 219 species (*System der analytischen Geometrie und der Theorie der Curven dritter Ordnung*, Berlin, 1835). His classification forms the subject of an exhaustive criticism by Cayley in the *Transactions of the Cambridge*

Philosophical Society for 1864, pp. 81–128, where the reader will find a comparison of the results with those of Newton.

80. Lastly, I may call attention to the classification of cubics given in Salmon's *Higher Plane Curves*, where a method not very different from that of Newton is adopted. See the second edition, Arts. 195–209.

81. In conclusion I may perhaps add that Waring applied the Newtonian method to quartic curves. Treated in this way, there are 12 characteristic forms of their equations, and not more than 84,551 species. (See his *Miscellanea Analytica*, Cambridge, 1762, Bk. II., chap. v.) Cramer divided quartics into 9 classes, but did not continue the subdivision into genera and species. Euler's method gives 8 classes subdivided into 146 genera. Salmon gives 10 classes.

APPENDICES.

82. The following extracts are literal transcripts of parts of two of the manuscripts alluded to in the text. No corrections are inserted, but (i.) words which are obviously redundant are placed in square brackets; (ii.) punctuation has been added; (iii.) the articles are numbered, for convenience of reference; and (iv.) the notation used to express a ratio has been altered from the obsolete form to that now current: that is, the ratio of a to b is printed here as $a : b$, while Newton and his contemporaries would have written it as $a . b$. Such notes as are given are in English and are placed in square brackets.

APPENDIX I.

Extracts from Newton's Manuscript entitled *Fragments concerning lines of the third order.*

(See Articles 11, 58–62, above.)

83. [The manuscript commences with a discussion of geometrical properties (especially of diameters) of conics and other curves (see Art. 58). After which, Newton proceeds to ennunciate the two propositions stated in Arts. 60, 61. The text is as follows.]

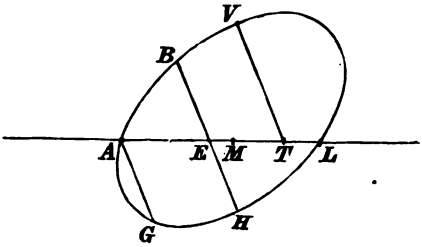
84. Caeterum cum hujusmodi proprietates eo praesertim spectent ut sciamus commode exprimere Curvas, eo ut de iisdem commode ratiocinemur dicam breviter qua ratione id fiat in Curvis cubicis aeque ac in Conicis sectionibus.

85. Sit primo ABC con. sectio. Eam secent rectae duae parallelae AG, BH in punctis A, G, B, H; et tertia quaevis recta non parallela AL in puncto L.

Fiat $$AG : EH-BE :: AM : EM,$$

et sit $$d : e :: AG : AM,$$

et $$b : e :: EB \times EH : EA \times EL.$$

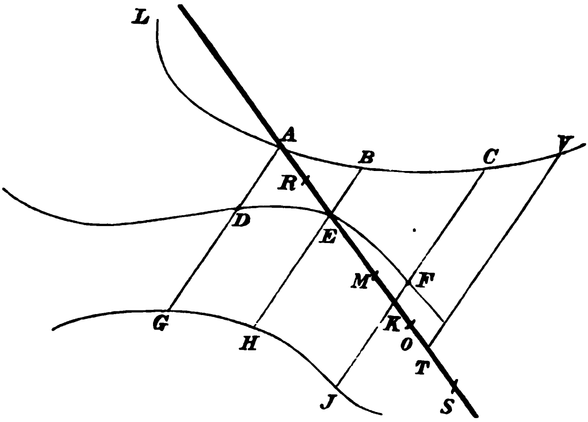


Et ad AE erecta quavis $TV \parallel BE$; et occurente curvae in V, dicatur $AE=y$ [an obvious slip for $TV=y$], et erit

$$yy-\frac{d}{e}TM\times y-\frac{b}{e}TA\times TL=0,$$

expressio conicae sectionis.

86. Eodem modo se res habet in curvis cubicis. Dentur enim positione tres rectae parellelae AG, BH, CJ secantes curvam propositam singulae in totidem datis punctis A, D, G; B, E, H; C, F, J. Et per extremum unius rectae punctum A et alterius medium punctum E agatur quarta recta AE, eaque producta secet curvam in



L et rectam CJ in K. Praeterea in AE capiatur punctum M ita ut sit

$$AD+AG : EH-EB :: AM : EM,$$

et sit

$$d : e :: AD+AG : AM,$$

[ut] et

$$b : e :: KC \times KF \times KJ : KL \times KA \times KE.$$

Deinde, facto

$$KJ : KF :: KC-KJ : KM,$$

capiantur etiam in AE puncta R et S ita ut sint

$$[AD \times AG : AR \times AS :: EB \times EH : ER \times ES],$$

$$AD \times AG : AR \times AS :: EB \times EH : ER \times ES$$

$$:: KC \times KF - KC \times KJ - KF \times KJ : KR \times KS :: e : c.$$

Et, acta quavis recta TV ipsi AG parallela, et occurrens [probably a slip for occurrente] rectae AE in T et curvae ABC in V, si TV dicatur y erit

$$y^3 - \frac{d}{e} \times TM \times yy + \frac{c}{e} TR \times - TS \times y - \frac{b}{e} TA \times TL \times TE = 0.$$

Estque generalis regula exprimendi curvas omnes ordinis cubici, si modo signorum + et − mutatio pro diversa positione linearum probe fiat.

87. Puncta autem R et S per planam Geometriam investigantur. Scilicet faciendum est

$$\frac{AD \times AG \times EK}{AE} + \frac{EB \times EH \times KA}{AE}$$

$$+ KC \times KJ - KC \times KF + KJ \times KF : AK \times EK :: b : e,$$

et erit $b : e :: AD \times DG : AR \times AS :: BE \times EH : ER \times ES.$

Dantur igitur rectangula

$$AR \times AS \quad \text{et} \quad ER \times ES.$$

Sint ista

$$AE \times p \quad \text{et} \quad AE \times q,$$

et erit

$$p+q+AE = AR+AS.$$

Ejus summae dimidium esto AO et erunt

$$TR \quad \text{et} \quad TS = \sqrt{AT^2 - AE \times p},$$

ubi AT^2 semper major erit quam $AE \times p$, si modo recta AE non per duo extrema linearum puncta A et B vel G et H, nec per media D et E, sed per extremum et medium uti A et E. Unde constat hanc regulam nulli exceptioni obnoxiam esse.

88. Caeterum cum haec sit aequatio cubica quae non nisi per solidam geometriam construi possit docebimus quomodo hae curvae redigendae sunt ad aequationes quae per planam Geometriam construi possunt.

Et in hunc finem sumamus plenam aliquam aequationem.

[Here follow the general equation of the third degree in x and y, and a discussion of the possible simplifications produced in it by transformation of coordinates. Much of this is crossed out, and possibly the whole should be cancelled as being merely rough notes of the calculations necessary for the following analysis, which I have already summarized in Art. 11.]

89. Sit basis $AB = z$,

ordinatim incedens $BC = v$.

Proponatur aequatio plena

$$av^3 + bzvv + czzv + dz^3 + evv + fzv + gzz + hv + kz + l = 0.$$

Construatur quomodocunque haec aequatio

$$a + bp + cpp + dp^3 = 0,$$

et invento p, cape BD ad BC ut p ad 1 [see Fig. 1], et agatur CD quae sit ad BC ut q ad 1. Et dicto

$$AD = r \text{ et } CD = s,$$

erit

$$BC = qs \text{ et } AB = r - pqs.$$

Quibus scriptis in aequatione pro v et z, orietur nova aequatio in qua terminus multiplicatus per s^3 deerit. Sit ista

$$brss + crrs + dr^3 + ess + frs + grr + hs + kr + l = 0.$$

[The analysis requires that $AB = r + pqs$. Hence, if p is positive, D must be between A and B, and not as in Newton's figure.]

90. Et si terminus $brss$ non desit produc [Fig. 1] CD ad E ut sit $2b : c :: AD : DE$. Et propter datos angulos trianguli ADE simul dabitur ratio AD ad AE; sit ista n ad 1. Et dictis

$$AE = t \text{ et } CE = w,$$

erit $$AD = nt \text{ et } CD = w - \frac{cn}{2b}t.$$

Quibus scriptis in aequatione pro r et s emerget aequatio sine

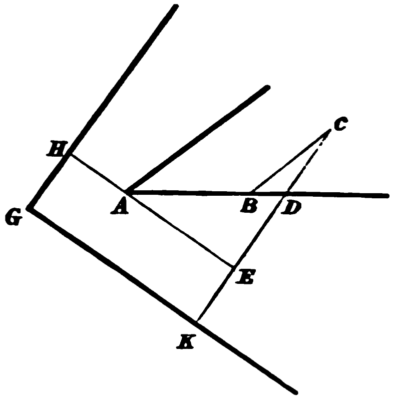


Fig. 1.

termino multiplicato per ttw. Sit ista

$$btww + eww + ftw + hw + dt^3 + gtt + kt + l = 0,$$

retentis denuo literis cum mutato significatu. Et produc CE ad K ut sit

$$EK = \frac{f}{b}, \quad \left[\text{a slip for } \frac{f}{2b}\right],$$

et EA ad H ut sit $$AH = \frac{e}{b},$$

et completo parallelogrammo $HEKG$, dic

$$GK = x \text{ et } CK = y,$$

et erit $$AE = x - \frac{e}{b} \text{ et } CE = y - \frac{f}{b} \quad \left[\text{it should be } y - \frac{f}{2b}\right].$$

Quibus scriptis pro t et w, orietur aequatio sine terminis yy et xy, exprimens relationem inter basem GK et ordinatam KC et non ultra reducibilis. Sit ista

$$bxyy + ky + dx^3 + gxx + kx + l = 0.$$

91. Quod, si terminus $brrs$ supra defuerit, non autem terminus $crrs$, fac $c : d :: AD : DE$, et pro r et s scriptis nt et $w - \frac{dn}{c}t$, emerget aequatio sine termino t^3. Sit ista

$$eww + cttw + ftw + hw + gtt + kt + l = 0.$$

Et positis $EK = \frac{g}{c}$ et $AH = \frac{f}{2c}$,

et pro t et w scripto $x - \frac{f}{2c}$ et $y - \frac{g}{c}$,

orietur aequatio sine terminis x^3, xyy, y^3, xx, xy, hujus formae

$$eyy + cxxy + hy + kx + l = 0$$

[quae quidem].

92. Sin terminus etiam $crrs$ defuerit non autem ess, fac

$$2e : f :: AD : DE,$$

et perinde, scriptis nt et $w - \frac{f}{2e}nt$ pro r et s, orietur aequatio sine terminis w^3, tww, ttw, tw. [Sit.] Deinde, positis

$$EK = \frac{h}{2e} \quad \text{et} \quad AH = \frac{g}{3d},$$

et pro t et w perinde scriptis

$$x - \frac{g}{3d} \quad \text{et} \quad y - \frac{h}{2e},$$

orietur aequatio hujus formae

$$eyy + dx^3 + kx + l = 0.$$

93. Quod si terminus ess etiam defuerit, non autem terminus frs, ... produc [Fig. 2] DA ad H ut sit

$$AH = \frac{h}{f}.$$

Et dicto DH t, erit $AD = t - \frac{h}{f}$.

Quo scripto pro r orietur aequatio hujus formae

$$fts + dt^3 + gtt + kt + l = 0.$$

Dein produc CD ad E et K ut sit

$$f : g :: HD : DE, \quad \text{et} \quad EK = \frac{k}{f}.$$

Et actâ HE comple parallelogrammum $HEKG$, sitque

$$HD : DE :: n : 1,$$

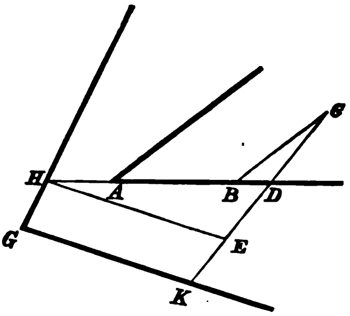


Fig. 2.

ut $$GK \text{ vel } HE = x \quad \text{et} \quad KC = y,$$

erit $$HD = nx, \quad \text{et} \quad CD = y - \frac{k}{f} - \frac{qnx}{f}.$$

Quibus scriptis pro t et s emerget aequatio hujus formae

$$fxy + dx^3 + l = 0.$$

94. Sin denique terminus etiam frs desit, fac

$$AH = \frac{g}{3d},$$

et scripto $t - \frac{g}{3d}$ pro r, orietur aequatio hujus formae

$$hs + dt^3 + kt + l = 0.$$

Deinde posito $$h : k :: HD : DE \quad \text{et} \quad EK = \frac{l}{h},$$

[illegible]sque nx pro t et $y - \frac{l}{h} - \frac{k}{h}nx$ pro s, orietur aequatio

$$hy + dn^3x^3 = 0.$$

95. Atque ita videre est quod omnes aequationes ad hasce quatuor

formas reduci possint

$$bxyy = dx^3+gxx+hy+kx+l\ [=0],$$

$$eyy = dx^3+kx+l\ [=0],$$

$$fxy = dx^3+l\ [=0],$$

et

$$hy = dx^3\ [=0],$$

ubi tamen signa + et — omnimodo variari possint et aliqui termini deesse. Atque hae formae non sunt ultra reducibiles.

96. [The remainder of the manuscript has been briefly described in Art. 62, and is as follows.]

97. Jam vero in prima harum formarum, posito $bxyy$ affirmativo, si terminus dx^3 sit etiam affirmativus curva erit Hyperbola triformis cum tribus Asymptotis, quarum nullae erunt parallelae. Sin termini isti [probably a slip for terminus iste] dx^3 sit negativus duae ex Hyperbolis mutabuntur in Ellipsin, quae tamen non semper habet ovalem formam, sed aliquando conjungitur reliquae figurae Hyperbolicae, aliquando evadit punctum, et aliquando imaginaria est. Inter Hyperbolam et Ellipsin media existit Parabola cum Hyperbola conjugata ubi dx^3 nullus est dummodo terminus etiam gxx [non] non desit. Nam si iste etiam desit, figura rursum evadet omni ex parte Hyperbolica. Et quidem triplex Hyperbola cum tribus Asymptotis quarum duae sunt parallelae si terminus k affirmativus est. Ubi vero k evanescit una Hyperbola evanescit et parallelae Asymptoti coincidunt, et ex his hyperbolis alia evanescit ubi k fit negativus.

98. In secunda forma curva semper Parabolica est sine aliqua Hyperbola conjugata et aliquando habet Ellipsin conjugatam juxta verticem. In tertia forma exprimit Parabolam Cartesianam. Et quarta Parabolam simplicem cubicam.

99. Atque ita patet novem esse genera curvarum ordinis cubici. Sub hisce tamen multae comprehenduntur species. Nam aliae non habent aliquam diametrum bissectionis; aliae habent, quas bifidas nominare licebit. Aliquae nullam habent partem similem alii parti; aliae habent partes omnino similes et aequales a centro intermedio, hinc inde infinitum procurrentes quas ideo aequicruras nominare licet. In aliquibus Asymptoti et omnes diametri bissectionis concurrunt ad idem punctum; in aliis non. Aliquae habent

punctum conjugatum, quas punctatas voco: aliae acutè terminantur ad vertices ad modum Cissoidis, quas ideo Cissoidales vel cuspidatas voco: et aliae se decussant ad modum crucis. Hae sunt praecipuae differentiae harum curvarum. Et secundum has differentias possumus enumerare species harum curvarum juxta sequentem tabulam.

[Here follows a tabular analysis of cubic curves into 7 classes, 14 genera, and 60 species. See Art. 62.]

Appendix II.

Extract from Newton's holograph manuscript, entitled *Enumeratio Curvarum Trium Dimensionum*.

(See Articles 39–43, 56, 57 above.)

100. [This extract is confined to the section on the Discrimination of Genera, which I described in Arts. 41–43 above. Newton's notes on the subject are written on a sheet of paper of a size slightly smaller than the rest of the manuscript, and with different ink. On the other hand, the absence of a division into articles, the absence of references, and the absence of marginal headings are points in which it is similar to this manuscript, and differs from all the others; and, as it has been preserved with the first manuscript, I treat it as a part of it. It runs as follows.]

101. Enumeratis harum Curvarum *speciebus* et ostenso quo pacto aequationes quibus *quomodocunque exprimuntur* reduci possint ad formulas quarum *subsidio definivimus ipsas* et distinximus in species videor determina*tionem horum locorum* ad fi*nem*[*] *per*duxisse. Sed quoniam reduct*io aequationum* ad *formulas istas* saepenumero taedio esse possit, ex abundanti jam *docebo* quomodo speciem curvae sine istius modi trans*mutatione diagnoscere* liceat. Supponamus igitur quae*stionem* aliquam *de* hujus *generis Loco* ad aequationem *per*ductam esse, sitque *aequatio* illa ut supra

$$y^3+bxyy+cxxy+dx^3+eyy+fxy+gxx+hy+kx+l=0,$$

in qua x denotat basem AB, y ordinatam BC, et $+b$, $+c$, $+d$, &c., datas quantitates cum suis signis affectas. Construatur haec aequatio

$$z^3+bzz+cz+d=0,$$

et ponatur

$$3\,[a]\,zz+2bz+c=m,\quad 3z+b=n,\quad ez^2+fz+g=p,$$

$$2ez+f=q\quad \text{et}\quad hz+k=r.$$

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[*] A hole has been burnt in the paper; and the letters in italics are conjecturally restored.

Ubi, si in construenda praefata aequatione tres obvenerint reales radices z, illae dabunt totidem quantitates m, n, p, q, r. Et si forte d non habeatur tunc duae radicum ex aequatione

$$zz+bz+c=0$$

elici debent, et 0 pro tertia usurpari; hoc est, pro m, n, p, q, r scribendum erit c, b, g, f et k, neglectis terminis in quibus z sive 0 reperitur. Eodem modo si c ac d simul desunt duae reputandae sunt radices et sibi mutuo et nihilo aequales et tertia radix erit $-b$. Et si omnes tres termini b, c, d desunt tres radices reputandae sunt aequales, et pro illis scribendum erit 0.

102. Jam igitur, si tres obvenerint inaequales radices, et nulla earum in colligendis quantitatibus m, n, p, q, r usurpata efficiat terminos $npp-qpm+rmm$ sese destruere, ffigura erit Hyperbola triformis non bifida. Sed si aliqua earum efficiat terminos istos $npp-qmm+rmm$ evanescere Hyperbola illa triformis erit uno modo bifida, et trifariam bifida si termini illi in unoquoque trium casuum evanescunt.

103. Quod si aequatio illa non nisi unicam habeat radicem z, caeteris duabus existentibus imaginariis, ffigura erit Hyperbola Elliptica modo non sit

$$nn-2m=0,$$

nam in hoc casu habebitur Hyperbolismus Ellipseos.

104. Ad haec si trium radicum z duae sunt aequales, et tertia usurpata pro z non efficiat

$$p=0,$$

ffigura erit Hyperbola Parabolica. Sed si efficiatur

$$p=0,$$

habebitur Hyperbolismus Hyperbolae, et Hyperbolismus Parabolae si insuper sit

$$qq-4nr=0.$$

Et in omnibus hisce casibus figura erit bifida si tertia illa radix z efficiat

$$npp-qpm+rmm=0;$$

aliter erit non bifida.

105. Denique si omnes tres radices z aequales esse contigerit, et non sit

$$p=0,$$

ffigura erit Parabola divergens. Sed si sit

$$p=0,$$

habebitur Parabolismus Hyperbolae nisi insuper sit

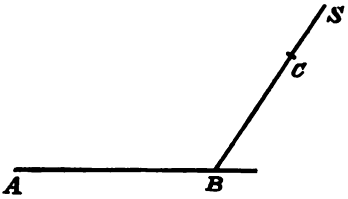
$$q=0,$$

in quo casu ffigura erit Parabola Cubica.

106. Cognita specie curvae si praeterea desideretur positio Asymptotorum et plaga ad quam crura Parabolica tendunt. In [a slip for , in] ordinata *BC*, quam y designat, cape

$$BS=zx-\frac{p}{m},$$

et punctum S attinget asymptoton si modo z non sit una duarum vel trium aequalium radicum. Hoc igitur pacto tot invenies Asymptotos quot sunt radices z non aequales. Sed si z sit una duarum aequalium radicum cape



$$BS=zx-\frac{q}{2n}\pm\sqrt{\frac{qq-4rn}{4mn}},$$

et habebis duas parallelas Asymptotos si figura sit Hyperbolismus Hyperbolae [aut aut] aut si existente

$$qq-4rn=0,$$

ffigura sit Hyperbolismus Parabolae, habebitur Asymptotos ponendo

$$BS=zx-\frac{q}{2n}.$$

Ac denique si tres sunt aequales radices z et figura sit Parabolismus Hyperbolae, Asymptotos determinabitur ponendo

$$BS=zx-\frac{r}{q}.$$

Eodem modo ubi curva est de genere quovis Parabolico, capiendo

$$BT=zx,$$

punctum T attingit lineam rectam quae tendat ad eandem plagam cum infinitis cruribus Parabolicis figurae, si modo pro z sumatur una aequalium radicum.

107. In praefatis determinationibus supposui primum terminum aequationis y^3 non deesse. Quamobrem si terminus ille desit et dx^3 non desit, debet y fieri basis figurae et x ordinata, et caetera peragi ut supra. Sed si uterque terminus simul desit sed b et c non desunt, figura erit Hyperbola triformis cujus una Asymptotos determinatur

capiendo $$AB = -\frac{e}{b},$$

et ad B erigendo lineam parallelam ordinatis, altera determinatur

capiendo $$BS = -\frac{g}{c}$$

et ducendo per S lineam parallelam basi. Nam istae parallelae erunt Asymptoti. Tertia attingetur a puncto S sumendo

$$BS = -\frac{cx}{b} + \frac{ec}{bb} - \frac{f}{b} + \frac{g}{c}.$$

Denique si terminorum etiam b et c alteruter puta c desit, figura vel Hyperbola Parabolica erit vel Hyperbolismus aliquis cujus determinationem supra satis explicuimus.

On the q-Series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ *and* $\frac{1}{4}K$. *By* J. W. L. GLAISHER, Sc.D., F.R.S.

[*Read Dec.* 11*th*, 1890.]

1. In a paper on the function $H(n)$,* which denotes the excess of the number of divisors of n which $\equiv 1$, mod. 3, over the number which $\equiv 2$, mod. 3, it was shown that the q-series having $H(n)$ as the coefficient of its general term, n denoting any integer, was expressible by means of a zeta function of argument $\frac{1}{3}K$, and that the q-series in which $H(m)$ was the coefficient of the general term, m denoting any uneven integer, was expressible by means of an elliptic function of $\frac{1}{4}K$. These results suggest that it would be of interest to obtain the developments in ascending powers of q of the complete system of

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* Vol. XXI., p. 395.

q-series which represent the sixteen elliptic and zeta functions of $\frac{1}{3}K$,* in order to determine the nature of the arithmetical functions which form their coefficients. The principal object of the present paper is to examine these arithmetical functions; but the elliptic functions of $\frac{1}{3}K$, the changes produced by the change of q into q^3, and other matters that arise in connexion with the q-series, are also considered. The concluding sections (§§ 39–56) relate to the elliptic and zeta functions of $\frac{1}{2}K$ and $\frac{1}{4}K$.

Developments of $k\rho \operatorname{sn} \frac{1}{3}K$, *&c., in ascending powers of* q. §§ 2–7.

2. Six of the sixteen functions of $\frac{1}{3}K$ are expressible by means of the function $E(n)$, which denotes the excess of the number of divisors of n which $\equiv 1$, mod. 4, over the number of divisors which $\equiv 3$, mod. 4. These expansions may be written:

$$k\rho \operatorname{sn} \tfrac{1}{3}K = k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty E(m)\, q^{\frac{1}{2}m} + 6\Sigma_1^\infty E(m)\, q^{\frac{3}{2}m},$$

$$kk'\rho \operatorname{sd} \tfrac{1}{3}K = k\rho \operatorname{cn} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{1}{2}m}$$
$$+ 6\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{3}{2}m},$$

$$\rho \operatorname{ns} \tfrac{1}{3}K = \rho \operatorname{dc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty E(n)\, q^n + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{ds} \tfrac{1}{3}K = k'\rho \operatorname{nc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty (-1)^n E(n)\, q^n + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

$$k'\rho \operatorname{nd} \tfrac{1}{3}K = \rho \operatorname{dn} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty E(n)\, q^n + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{dn} \tfrac{1}{3}K = k'\rho \operatorname{nd} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty (-1)^n E(n)\, q^n + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

where ρ denotes $\dfrac{2K}{\pi}$, and m and n denote, as throughout this paper, any uneven number, and any number, respectively.

3. Five others are expressible by means of $H(n)$, defined as in § 1, and by $H'(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$, mod. 3, and have uneven conjugates, over the number of divisors which $\equiv 2$, mod. 3, and have uneven conjugates. These expansions are

$$k\rho \operatorname{cd} \tfrac{1}{3}K = k\rho \operatorname{sn} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty H(m)\, q^{\frac{1}{2}m},$$

$$k\rho \operatorname{cn} \tfrac{1}{3}K = kk'\rho \operatorname{sd} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m)\, q^{\frac{1}{2}m},$$

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* The complete system of q-series for $k\rho \operatorname{sn} u$, &c., is given in the *Messenger of Mathematics*, Vol. XVII., pp. 2, 3. The four zeta functions $\operatorname{zn} u$, $\operatorname{zs} u$, $\operatorname{zc} u$, $\operatorname{zd} u$ are there denoted by $Z(u)$, $Z_1(u)$, $Z_2(u)$, $Z_3(u)$.

$$-\rho\,\mathrm{zd}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{zn}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty H'(n)\,q^n,$$

$$\rho\,\mathrm{zn}\,\tfrac{1}{3}K = -\rho\,\mathrm{zd}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty (-1)^{n-1} H'(n)\,q^n,$$

$$-\rho\,\mathrm{zc}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1+6\Sigma_1^\infty H(n)\,q^{2n}\right\}.$$

4. Two of the elliptic functions of $\frac{1}{3}K$ are expressible by means of a function $J(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$, mod. 6, over the number of divisors which $\equiv 5$, mod. 6; viz., we have

$$\rho\,\mathrm{dc}\,\tfrac{1}{3}K = \rho\,\mathrm{ns}\,\tfrac{2}{3}K = \frac{2}{\sqrt{3}}\left\{1+3\,\Sigma_1^\infty J(n)\,q^n\right\},$$

$$k'\rho\,\mathrm{nc}\,\tfrac{1}{3}K = \rho\,\mathrm{ds}\,\tfrac{2}{3}K = \frac{2}{\sqrt{3}}\left\{1+3\,\Sigma_1^\infty (-1)^n J(n)\,q^n\right\}.$$

When n is uneven, $J(n)$ is evidently equal to $H(n)$, so that in the first two formulæ of § 3 we may replace $H(m)$ by $J(m)$. These formulæ may, therefore, be written:

$$k\rho\,\mathrm{cd}\,\tfrac{1}{3}K = \quad k\rho\,\mathrm{sn}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty J(m)\,q^{\frac{1}{2}m},$$

$$k\rho\,\mathrm{cn}\,\tfrac{1}{3}K = kk'\rho\,\mathrm{sd}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} J(m)\,q^{\frac{1}{2}m}.$$

5. Denoting by $I(n)$ the excess of the number of divisors of n which $\equiv 1$ or 2, mod. 6, over the number of divisors which $\equiv 4$ or 5, mod. 6, we find

$$\rho\,\mathrm{zs}\,\tfrac{1}{3}K = -\rho\,\mathrm{zc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1+2\,\Sigma_1^\infty I(n)\,q^{2n}\right\}.$$

If n be uneven,

$$I(n) = J(n);$$

if n be even,

$$I(n) = J(n)+H(\tfrac{1}{2}n).$$

We may therefore express this formula also by means of the functions J and H as follows:

$$\rho\,\mathrm{zs}\,\tfrac{1}{3}K = -\rho\,\mathrm{zc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1+2\,\Sigma_1^\infty J(n)\,q^{2n}+2\Sigma_1^\infty H(n)\,q^{4n}\right\}.$$

6. The remaining two functions, sc and cs, may be expressed by the formulæ

$$k'\rho\,\mathrm{sc}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{cs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1-6\,\Sigma_1^\infty \mathrm{h}(n)\,q^{2n}\right\},$$

$$\rho\,\mathrm{cs}\,\tfrac{1}{3}K = k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1-2\,\Sigma_1^\infty \mathrm{i}(n)\,q^{2n}\right\},$$

where

$$\mathrm{h}(n) = H'(n) - H''(n),$$

$$\mathrm{i}(n) = I'(n) - I''(n).$$

The function $H'(n)$ has been defined in § 3; and $H''(n)$ is defined as the excess of the number of divisors of n which $\equiv 1$, mod. 3, and have even conjugates, over the number of divisors which $\equiv 2$, mod. 3, and have even conjugates. The functions $I'(n)$ and $I''(n)$ stand to $I(n)$ in the same relation as $H'(n)$ and $H''(n)$ to $H(n)$.*

7. The arithmetical functions, which serve to express the coefficients in the sixteen series, are, therefore, seven in number, viz.:

$$E(n), \quad H(n), \quad H'(n), \quad H''(n), \quad J(n), \quad I(n), \quad \mathrm{i}(n);$$

but the first of these, $E(n)$, occurs in the series for ρ, $k\rho$, $k'\rho$, &c., in ascending powers of q, and is not specially connected with the argument $\frac{1}{3}K$.

When n is uneven, $H''(n)$ is zero, and all the other functions become equal, so that

$$E(m) = H(m) = H'(m) = J(m) = I(m) = \mathrm{i}(m).$$

The Functions H, H′, H″, i. §§ 8, 9.

8. Let

$$n = 2^r m,$$

m being an uneven number, and let $1, a, b, \dots, m$ be all the divisors of m which $\equiv 1$ or 2, mod. 3; these are the divisors upon which the value of $H(m)$ depends. Now, in $2^r m$, the divisors which have uneven conjugates are $2^r, 2^r a, 2^r b, \dots 2^r m$; *i.e.*, they are the former divisors each multiplied by 2^r. Now, by multiplying a divisor $\equiv 1$, mod. 3, by 2, we produce a divisor $\equiv 2$, mod. 3, and *vice versâ*. Thus, evidently,

$$H'(2m) = -H'(m), \quad H'(4m) = H(m),$$

and in general

$$H'(n) = (-1)^r H(m).$$

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* In the case of the functions H and I, the single accent indicates that in forming the function only the divisors whose conjugates are uneven are to be taken into account, and the double accent indicates that only divisors whose conjugates are even are to be taken into account. The accent in $\Delta'(n)$ (§ 31) has a different meaning.

9. If r be uneven, $H(n) = 0$;

and if r be even, $H(n) = H(m)$.

We may therefore write

$$H(n) = \tfrac{1}{2}\left\{1+(-1)^r\right\} H(m);$$

and, since $H'(n)+H''(n) = H(n)$,

we find $H''(n) = \tfrac{1}{2}\left\{1-(-1)^r\right\} H(m)$.

This latter result may be independently established without difficulty; for the divisors of n which $\equiv 1$ or 2, mod. 3, and have even conjugates, are $1, a, b, \dots, m$; $2, 2a, 2b, \dots 2m$; ...; $2^{r-1}, 2^{r-1}a, 2^{r-1}b, \dots 2^{r-1}m$. Thus there are r systems of such divisors. The divisors in each system are opposite in character to those in the preceding system; so that, if

$$r = 2, \quad H''(n) = 0;$$

if $r = 3, \quad H''(n) = H(m)$;

and, in general, $H''(n) = 0$ or $H(m)$,

according as r is even or uneven.

We have also

$$\mathrm{h}(n) = H'(n)-H''(n) = \tfrac{1}{2}\left\{3(-1)^r-1\right\} H(m).$$

The Functions J, I, I′, i. §§ 10, 11.

10. If, as before, we put

$$n = 2^r m,$$

we have $J(n) = H(m)$.

Since (§ 5), if n be even,

$$I(n) = J(n)+H(\tfrac{1}{2}n),$$

and (§ 9), $H(\tfrac{1}{2}n) = \tfrac{1}{2}\left\{1-(-1)^r\right\} H(m)$,

we find

$$I(n) = H(m)+\tfrac{1}{2}\left\{1-(-1)^r\right\} H(m) = \tfrac{1}{2}\left\{3-(-1)^r\right\} H(m).$$

11. It is easy to see that, if n be even,

$$I'(n) = H'(\tfrac{1}{2}n),$$

and
$$I''(n) = J(n) + H''(\tfrac{1}{2}n).$$

We thus find
$$I'(n) = (-1)^{r-1} H(m) = -H'(n),$$
$$I''(n) = \tfrac{1}{2}\{3+(-1)^r\} H(m), \quad \mathrm{i}(n) = -\tfrac{3}{2}\{1+(-1)^r\} H(m).$$

The complete set of Functions. §§ 12, 13.

12. It thus appears that all the H, J and I functions of n admit of being expressed in terms of $H(m)$, where m is the largest uneven divisor of n. Arranging the formulæ in one group, we have, if
$$n = 2^r m,$$
then
$$J(n) = H(m),$$
$$H(n) = \tfrac{1}{2}\{1+(-1)^r\} H(m),$$
$$H'(n) = (-1)^r H(m),$$
$$H''(n) = \tfrac{1}{2}\{1-(-1)^r\} H(m),$$
$$\mathrm{h}(n) = \tfrac{1}{2}\{3(-1)^r - 1\} H(m) = (-1)^r I(n),$$
$$I(n) = \tfrac{1}{2}\{3-(-1)^r\} H(m) = (-1)^r \mathrm{h}(n),$$
$$I'(n) = (-1)^{r-1} H(m) = -H'(n),$$
$$I''(n) = \tfrac{1}{2}\{3+(-1)^r\} H(m);$$
$$\mathrm{i}(n) = -\tfrac{3}{2}\{1+(-1)^r\} H(m) = -3H(n),$$
it being supposed, in the values of $I'(n)$, $I''(n)$, $\mathrm{i}(n)$, that n is even, *i.e.*, that r is not zero. If, therefore, r be even, then
$$H(n) = H'(n) = \mathrm{h}(n) = I(n) = -I'(n) = H(m), \quad H''(n) = 0,$$
$$I''(n) = 2H(m), \quad \mathrm{i}(n) = -3H(m);$$
if r be uneven, then
$$-H'(n) = H''(n) = I'(n) = I''(n) = H(m),$$
$$-\mathrm{h}(n) = I(n) = 2H(m), \quad H(n) = 0, \quad \mathrm{i}(n) = 0,$$
and (§ 7), in the case of an uneven number m,
$$H(m) = H'(m) = \mathrm{h}(m) = I(m) = I'(m) = \mathrm{i}(m) = H(m),$$
$$H''(m) = 0, \quad I''(m) = 0;$$
while, for all values of n,
$$J(n) = H(m).$$

13. Since i (n) is equal to $H(n)$ when n is even, and is equal to $-3H(n)$ when n is even, we have, for all values of n,

$$\mathrm{i}(n) = -\left\{1+2(-1)^n\right\} H(n).$$

In the group of formulæ at the beginning of the preceding section, all the functions were exhibited in terms of the exponent r and a single function $H(m)$. In such a mode of expression, it is perhaps preferable to take $J(n)$ instead of $H(m)$ as the single function. The two functions are equal as regards numerical value, but the former is more fundamental in definition and depends upon n instead of m.

The Series for $\rho \operatorname{cs} \frac{1}{3}K$, $k'\rho \operatorname{sc} \frac{1}{3}K$, $\rho \operatorname{zs} \frac{1}{3}K$. §§ 14, 15.

14. By substituting for i (n) its value in terms of $H(n)$, given in the preceding section, we obtain for $\rho \operatorname{cs} \frac{1}{3}K$ (§ 6) the formula

$$\rho \operatorname{cs} \tfrac{1}{3}K = k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3}\left\{1-2\Sigma_1^\infty H(m)\, q^{2m}+6\Sigma_1^\infty H(2n)\, q^{4n}\right\},$$

or, as we may write it,

$$\rho \operatorname{cs} \tfrac{1}{3}K = k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3}\left[1+2\Sigma_1^\infty \left\{1+2(-1)^n\right\} H(n)\, q^{2n}\right].$$

15. Since $\qquad \mathrm{h}(n) = 2H'(n)-H(n),$

and $\qquad I(n) = 3H(n)-2H'(n),$

we may express the series for $k'\rho \operatorname{sc} \frac{1}{3}K$ and $\rho \operatorname{zs} \frac{1}{3}K$ by means of the functions H and H', in the forms

$$k'\rho \operatorname{sc} \tfrac{1}{3}K = \quad \rho \operatorname{cs} \tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left[1-6\Sigma_1^\infty \left\{2H'(n)-H(n)\right\} q^{2n}\right],$$

$$\rho \operatorname{zs} \tfrac{1}{3}K = -\rho \operatorname{zc} \tfrac{2}{3}K = \sqrt{3}\left[1+2\Sigma_1^\infty \left\{3H(n)-2H'(n)\right\} q^{2n}\right].$$

The Functions H and J. § 16.

16. It may be observed that, since

$$H'(n) = 2H(n)-J(n),$$

$$\mathrm{h}(n) = 3H(n)-2J(n),$$

$$I(n) = -H(n)+2J(n),$$

we may express all the sixteen elliptic and zeta functions of $\frac{1}{3}$K as series of powers of q by means of the three functions E, H, J. Six of the formulæ involve E only, six involve H or J only, and four involve both H and J.

The six formulæ which involve E. §§ 17, 18.

17. Since

$$4\Sigma_1^\infty E(m)\, q^{\frac12 m} = 4\Sigma_1^\infty (-1)^{\frac12(m-1)} E(m)\, q^{\frac12 m} = k\rho,$$

and $$1+4\Sigma_1^\infty E(n)\, q^n = \rho, \quad 1+4\Sigma_1^\infty (-1)^n E(n)\, q^n = k'\rho,$$

it follows from the formulæ of § 2 that

$$k\rho \operatorname{sn} \tfrac13 K - \tfrac12 k\rho = 6\Sigma_1^\infty E(m)\, q^{\frac32 m},$$

$$-kk'\rho \operatorname{sd} \tfrac13 K + \tfrac12 k\rho = 6\Sigma_1^\infty (-1)^{\frac12(m-1)} E(m)\, q^{\frac32 m},$$

$$\rho \operatorname{ns} \tfrac13 K - \tfrac12 \rho = \tfrac32 + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{ds} \tfrac13 K - \tfrac12 k'\rho = \tfrac32 + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

$$k'\rho \operatorname{nd} \tfrac13 K + \tfrac12 \rho = \tfrac32 + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{dn} \tfrac13 K + \tfrac12 k'\rho = \tfrac32 + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n}.$$

18. If we denote by k_3, k'_3, ρ_3 the quantities into which k, k', ρ are converted by the change of q into q^3, these equations show that

$$k\rho \operatorname{sn} \tfrac13 K = \tfrac12 k\rho + \tfrac32 k_3\rho_3,$$

$$kk'\rho \operatorname{sd} \tfrac13 K = \tfrac12 k\rho - \tfrac32 k_3\rho_3,$$

$$\rho \operatorname{ns} \tfrac13 K = \tfrac12 \rho + \tfrac32 \rho_3,$$

$$\rho \operatorname{ds} \tfrac13 K = \tfrac12 k'\rho + \tfrac32 k'_3\rho_3,$$

$$k'\rho \operatorname{nd} \tfrac13 K = -\tfrac12 \rho + \tfrac32 \rho_3,$$

$$\rho \operatorname{dn} \tfrac13 K = -\tfrac12 k'\rho + \tfrac32 k'_3\rho_3.$$

Elliptic Functions of $\frac13 K$. §§ 19–24.

19. In the twelfth volume[*] of the *Messenger of Mathematics*, Mr. Burnside has given, in a very interesting form, the values of $\operatorname{sn}^2 \frac13 K$,

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[*] "The Elliptic Functions of $\frac13 K$, &c.," p. 154.

$\operatorname{sn}^2 \frac{1}{3}iK'$, &c. Mr. Burnside's formulæ lead directly to the following results:—

$$\operatorname{cd}^2 \tfrac{1}{3}K = \frac{1}{k}\left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1+\lambda}\right\},$$

$$\operatorname{cn}^2 \tfrac{1}{3}K = \frac{k'}{k}\left\{\sqrt{2(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu}\mp\sqrt{\mu-1}\right\},$$

$$\operatorname{cs}^2 \tfrac{1}{3}K = k'\left\{\sqrt{2(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}\right\},$$

where $$\lambda = \left(\frac{k'^2}{2k}\right)^{\frac{2}{3}}, \quad \mu = \left(\frac{1}{2kk'}\right)^{\frac{2}{3}}, \quad \nu = \left(\frac{k^2}{2k'}\right)^{\frac{2}{3}}.$$

In the formula for $\operatorname{cn}^2 \frac{1}{3}K$, the upper or lower sign is to be taken according as $k <$ or $> k'$.

20. Applying Jacobi's general formulæ of transformation to the case $n = 3$, we find

$$k_3 = k^3 \operatorname{sn}^4 \tfrac{1}{3}K, \qquad k_3' = \frac{1}{k'}\operatorname{dn}^4 \tfrac{1}{3}K,$$

$$M = \frac{\operatorname{sn}^2 \frac{1}{3}K}{\operatorname{sn}^2 \frac{2}{3}K} = \operatorname{sn}^2 \tfrac{1}{3}K \operatorname{dc}^2 \tfrac{1}{3}K, \qquad \frac{1}{M} = 3\,\frac{\rho_3}{\rho}.$$

These formulæ give

$$3\,\frac{\rho_3}{\rho} = \operatorname{ns}^2 \tfrac{1}{3}K \operatorname{cd}^2 \tfrac{1}{3}K, \quad \frac{k_3^{\frac{1}{2}}}{k^{\frac{1}{2}}} = k \operatorname{sn}^2 \tfrac{1}{3}K, \quad \frac{k_3'^{\frac{1}{2}}}{k'^{\frac{1}{2}}} = \frac{1}{k'}\operatorname{dn}^2 \tfrac{1}{3}K,$$

whence $$3\,\frac{k_3^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}\rho} = k \operatorname{cd}^2 \tfrac{1}{3}K, \quad 3\,\frac{k_3'^{\frac{1}{2}}\rho_3}{k'^{\frac{1}{2}}\rho} = \frac{1}{k'}\operatorname{cs}^2 \tfrac{1}{3}K.$$

Substituting the values of $\operatorname{cd}^2 \frac{1}{3}K$ and $\operatorname{cs}^2 \frac{1}{3}K$ from the last section, we find

$$3\,\frac{k_3^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}\rho} = \sqrt{2(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1+\lambda},$$

$$3\,\frac{k_3'^{\frac{1}{2}}\rho_3}{k'^{\frac{1}{2}}\rho} = \sqrt{2(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}.$$

21. By the change of q into q^2, $k^{\frac{1}{2}}\rho$ is converted into $\frac{1}{2}k\rho$, and $\frac{k'^2}{2k}$ into $\frac{2k'}{k^2}$; and by the change of q into $q^{\frac{1}{2}}$, $k'^{\frac{1}{2}}\rho$ is converted into $k'\rho$, and $\frac{k^2}{2k'}$ into $\frac{2k}{k'^2}$. By making these changes in the two formulæ,

respectively, we find

$$3\,\frac{k_3\rho_3}{k\rho} = \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} - \sqrt{1+u},$$

$$3\,\frac{k_3'\rho_3}{k'\rho} = \sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v} + \sqrt{1+v},$$

where
$$u = \left(\frac{2k'}{k^2}\right)^{\frac{2}{3}}, \qquad v = \left(\frac{2k}{k'^2}\right)^{\frac{2}{3}}.$$

By changing the sign of q, we convert $k'\rho$ into ρ, and $\dfrac{2k}{k'^2}$ into $2ikk'$. The second formula, therefore, gives

$$3\,\frac{\rho_3}{\rho} = \sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w} \pm \sqrt{1-w},$$

where
$$w = (2kk')^{\frac{2}{3}}.$$

The upper or lower sign is to be taken according as $k <$ or $> k'$ (§ 23).

22. The formulæ of § 18 may be written in two groups, as follows:

$$3\,\frac{k_3\rho_3}{k\rho} = 2\,\mathrm{sn}\,\tfrac{1}{3}K - 1,$$

$$3\,\frac{k_3'\rho_3}{k'\rho} = \frac{2}{k'}\,\mathrm{ds}\,\tfrac{1}{3}K - 1,$$

$$3\,\frac{\rho_3}{\rho} = 2\,\mathrm{ns}\,\tfrac{1}{3}K - 1\,;$$

$$\mathrm{sn}\,\tfrac{1}{3}K + k'\,\mathrm{sd}\,\tfrac{1}{3}K = 1,$$

$$\mathrm{ds}\,\tfrac{1}{3}K - \mathrm{dn}\,\tfrac{1}{3}K = k',$$

$$\mathrm{ns}\,\tfrac{1}{3}K - k'\,\mathrm{nd}\,\tfrac{1}{3}K = 1.$$

Of the three formulæ in the second group, any two are deducible at sight from the third.

23. By combining the results given in the two preceding sections, we obtain the following system of formulæ for six of the elliptic functions of $\tfrac{1}{3}K$:—

$$\mathrm{sn}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{1 + \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} - \sqrt{1+u}\right\},$$

$$k'\,\mathrm{sd}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{1 - \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} + \sqrt{1+u}\right\},$$

$$\mathrm{ns}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{\;\;1+\sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w}\pm\sqrt{1-w}\right\},$$

$$k'\,\mathrm{nd}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{-1+\sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w}\pm\sqrt{1-w}\right\},$$

$$\frac{1}{k'}\,\mathrm{ds}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{\;\;1+\sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v}+\sqrt{1+v}\right\},$$

$$\frac{1}{k'}\,\mathrm{dn}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{-1+\sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v}+\sqrt{1+v}\right\};$$

where $$u = \left(\frac{2k'}{k^2}\right)^{\frac{2}{3}},\quad w = (2kk')^{\frac{2}{3}},\quad v = \left(\frac{2k}{k'^2}\right)^{\frac{2}{3}}.$$

In the two middle formulæ, the upper or lower sign is to be taken according as $k <$ or $> k'$, as in the second of Mr. Burnside's formulæ. That this must be so is evident on putting $k = 0$ and $k = 1$. In both cases w vanishes; but, when $k = 0$, the value of $\mathrm{ns}\,\frac{1}{3}K$ is 2, and when $k = 1$ it is unity.

24. In the volume of the *Messenger*[*] already referred to (§ 19), Dr. Forsyth has obtained the value of $\mathrm{sn}\,\frac{1}{3}K$ in the form

$$\frac{2}{1+(1-c)^{\frac{1}{2}}+\left\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}},$$

where $$c^3 = 4k^2k'^2;$$

the sign of the radical $(1-c)^{\frac{1}{2}}$ having been determined by putting $k = 0$. This result is equivalent to

$$2\,\mathrm{ns}\,\tfrac{1}{3}K = 1+(1-c)^{\frac{1}{2}}+\left\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

which is the same as the third of the above equations, when $k < k'$.

System of values of ρ_3, $k_3\rho_3$, &c. § 25.

25. From § 20, we find at once, by multiplication,

$$3\,\frac{k_3^{\frac{1}{2}}k_3'^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho} = \frac{k}{k'}\,\mathrm{cn}^2\,\tfrac{1}{3}K;$$

and, by combining this equation with Mr. Burnside's second formula

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[*] "The Elliptic Functions of $\frac{1}{3}K$," Vol. XII., p. 134.

(§19), and collecting all the results into one group, we have

$$3\,\frac{k_3^{\frac12}\rho_3}{k^{\frac12}\rho}=\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda}-\sqrt{1+\lambda}=k\,\mathrm{cd}^2\,\tfrac13K,$$

$$3\,\frac{k_3^{\frac12}k_3'^{\frac12}\rho_3}{k^{\frac12}k'^{\frac12}\rho}=\sqrt{2(1+\mu+\mu^2)^{\frac12}-2-\mu}\mp\sqrt{\mu-1}=\frac{k}{k'}\,\mathrm{cn}^2\,\tfrac13K,$$

$$3\,\frac{k_3'^{\frac12}\rho_3}{k'^{\frac12}\rho}=\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu}+\sqrt{1+\nu}=\frac{1}{k'}\,\mathrm{cs}^2\,\tfrac13K,$$

$$3\,\frac{k_3\rho_3}{k\rho}=\sqrt{2\,(1-u+u^2)^{\frac12}+2-u}-\sqrt{1+u}$$

$$=k^2\,\mathrm{sn}^2\,\tfrac13K\,\mathrm{cd}^2\,\tfrac13K=2\,\mathrm{sn}\,\tfrac13K-1,$$

$$3\,\frac{\rho_3}{\rho}=\sqrt{2\,(1+w+w^2)^{\frac12}+2+w}\pm\sqrt{1-w}$$

$$=\mathrm{ns}^2\,\tfrac13K\,\mathrm{cd}^2\,\tfrac13K=2\,\mathrm{ns}\,\tfrac13K-1,$$

$$3\,\frac{k_3'\rho_3}{k'\rho}=\sqrt{2\,(1-v+v^2)^{\frac12}+2-v}+\sqrt{1+v}$$

$$=\frac{1}{k'^2}\,\mathrm{ds}^2\,\tfrac13K\,\mathrm{cn}^2\,\tfrac13K=\frac{2}{k'}\,\mathrm{ds}\,\tfrac13K-1;$$

where
$$\lambda=\left(\frac{k'^2}{2k}\right)^{\frac13},\quad \mu=\left(\frac{1}{2kk'}\right)^{\frac13},\quad \nu=\left(\frac{k^2}{2k'}\right)^{\frac13},$$

$$u=\left(\frac{2k'}{k^2}\right)^{\frac13},\quad w=(2kk')^{\frac13},\quad v=\left(\frac{2k}{k'^2}\right)^{\frac13},$$

so that
$$\lambda v=1,\quad \mu w=1,\quad \nu u=1.$$

The case of $k=\dfrac{1}{\sqrt2}$. §26.

26. Putting $k=k'=\dfrac{1}{\sqrt2}$ in the formulæ for $\mathrm{sn}\,\tfrac13K$, &c., in §23, we find that, in this case,

$$\mathrm{sn}\,\tfrac13K=\frac{1-3^{\frac12}+2^{\frac12}3^{\frac14}}{2},$$

$$\mathrm{dn}\,\tfrac13K=\frac{-1+3^{\frac12}+2^{\frac12}3^{\frac14}}{2^{\frac32}},$$

$$\operatorname{sd}\tfrac{1}{3}K = \frac{1+3^{\frac12}-2^{\frac12}3^{\frac14}}{2^{\frac12}},$$

$$\operatorname{ds}\tfrac{1}{3}K = \frac{1+3^{\frac12}+2^{\frac12}3^{\frac14}}{2^{\frac32}},$$

$$\operatorname{ns}\tfrac{1}{3}K = \frac{2^{\frac12}+3^{\frac14}+3^{\frac34}}{2^{\frac32}},$$

$$\operatorname{nd}\tfrac{1}{3}K = \frac{-2^{\frac12}+3^{\frac14}+3}{2}.$$

The values of $\operatorname{ns}\frac{1}{3}K$ and $\frac{1}{\surd 2}\operatorname{nd}\frac{1}{3}K$ present themselves in the forms

$$\tfrac{1}{2}\left\{1+(2\surd 3+3)^{\frac12}\right\} \quad\text{and}\quad \tfrac{1}{2}\left\{-1+(2\surd 3+3)^{\frac12}\right\},$$

which are reducible to the expressions given above by observing that

$$(2\surd 3+3)^{\frac12} = \frac{3^{\frac14}+3^{\frac34}}{2^{\frac12}}.$$

I had obtained the numerical values corresponding to the case of

$$k = \frac{1}{\surd 2}$$

before finding the general formulæ in § 23, so that the former afford a verification of the latter.

Systems of q-series involving E. §§ 27–29.

27. From § 17, we have

$$1+4\Sigma_1^\infty E(n)\,q^{3n} = \tfrac{1}{3}\left(2\operatorname{ns}\tfrac{1}{3}K-1\right)\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^{3n} = \tfrac{1}{3}\left(2\operatorname{ds}\tfrac{1}{3}K-k'\right)\rho,$$

$$4\Sigma_1^\infty E(m)\,q^{\frac32 m} = \tfrac{1}{3}\left(2\operatorname{sn}\tfrac{1}{3}K-1\right)k\rho;$$

in which equations the values of the coefficients of ρ and $k\rho$ in terms of k and k' may be at once written down from § 23.

28. The corresponding formulæ, in which q^3 is replaced by q, q^2 or q^4, are easily obtained; and, combining them with the above results,

we have the following groups of equations :—

$$1+4\Sigma_1^\infty E(n)\,q^n = \rho,$$

$$1+4\Sigma_1^\infty E(n)\,q^{2n} = \tfrac{1}{2}(1+k')\,\rho,$$

$$1+4\Sigma_1^\infty E(n)\,q^{3n} = \tfrac{1}{3}(2\,\mathrm{ns}\,\tfrac{1}{3}K-1)\,\rho,$$

$$1+4\Sigma_1^\infty E(n)\,q^{4n} = \tfrac{1}{4}(1+k'^{\frac{1}{2}})^2\rho\,;$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^n = k'\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^{2n} = k'^{\frac{1}{2}}\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^{3n} = \tfrac{1}{3}(2\,\mathrm{ds}\,\tfrac{1}{3}K-k')\,\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^{4n} = \frac{1}{\sqrt{2}}k'^{\frac{1}{4}}(1+k')^{\frac{1}{2}}\rho\,;$$

$$4\Sigma_1^\infty E(m)\,q^{\frac{1}{2}m} = k\rho,$$

$$4\Sigma_1^\infty E(m)\,q^{m} = \tfrac{1}{2}(1-k')\,\rho,$$

$$4\Sigma_1^\infty E(m)\,q^{\frac{3}{2}m} = \tfrac{1}{3}(2\,\mathrm{sn}\,\tfrac{1}{3}K-1)\,k\rho,$$

$$4\Sigma_1^\infty E(m)\,q^{2m} = \tfrac{1}{4}(1-k'^{\frac{1}{2}})^2\rho.$$

29. The three kinds of q-series may be expressed as series of the form $\Sigma_1^\infty \phi(q^n)$ by means of the identical equations :—

$$\Sigma_1^\infty E(n)\,q^n = \Sigma_1^\infty \frac{q^n}{1+q^{2n}} = \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\frac{q^m}{1-q^m},$$

$$\Sigma_1^\infty (-1)^{n-1} E(n)\,q^n = \Sigma_1^\infty (-1)^{n-1}\frac{q^n}{1+q^{2n}} = \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\frac{q^m}{1+q^m},$$

$$\Sigma_1^\infty E(m)\,q^m = \Sigma_1^\infty \frac{q^m}{1+q^{2m}} = \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\frac{q^m}{1-q^{2m}}.$$

Taking the first of the two forms, we may write

$$1+4\Sigma_1^\infty E(n)\,q^n = \Sigma_{-\infty}^\infty \operatorname{sech} na,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^n = \Sigma_{-\infty}^\infty (-1)^n \operatorname{sech} na,$$

$$4\Sigma_1^\infty E(m)\,q^m = \Sigma_{-\infty}^\infty \operatorname{sech} ma\,;$$

where $$a = \frac{\pi K'}{K}.$$

The case of $k = \frac{1}{\sqrt{2}}$. § 30.

30. Denoting $K^\circ\sqrt{2}$ by ω, where K° is the value of K when

$$k = \frac{1}{\sqrt{2}},$$

so that ω is the quantity so designated by Gauss in his posthumous papers on the Lemniscate functions,* we find

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} n\pi = 2^{\frac{1}{2}} \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 2n\pi = (2^{-\frac{1}{2}} + 2^{-1}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 3n\pi = (3^{-\frac{1}{4}} + 3^{-\frac{3}{4}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 4n\pi = (2^{-\frac{3}{4}} + 2^{-2} + 2^{-\frac{3}{2}}) \frac{\omega}{\pi};$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} n\pi = \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 2n\pi = 2^{\frac{1}{4}} \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 3n\pi = (2^{\frac{1}{2}} 3^{-\frac{3}{4}} + 3^{-\frac{1}{2}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 4n\pi = 2^{-\frac{3}{8}} (1 + 2^{\frac{1}{2}})^{\frac{1}{2}} \frac{\omega}{\pi};$$

and
$$\Sigma_{-\infty}^{\infty} \operatorname{sech} \tfrac{1}{2}m\pi = \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} m\pi = (2^{-\frac{1}{2}} - 2^{-1}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} \tfrac{3}{2}m\pi = (2^{\frac{1}{2}} 3^{-\frac{3}{4}} - 3^{-\frac{1}{2}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 2m\pi = (2^{-\frac{3}{4}} + 2^{-2} - 2^{-\frac{3}{2}}) \frac{\omega}{\pi}.$$

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* Gauss, *Werke*, Vol. III., p. 413.

Developments of $k^2\rho^2\,\mathrm{sn}^2\,\frac{1}{3}K$, &c., in ascending powers of q. §§ 31–33.

31. The q-series for the squared elliptic functions of $\frac{1}{3}K$ do not lead to any new arithmetical functions, being expressible by means of the functions $\Delta'(n)$, $\zeta(n)$, $\sigma(n)$ which occur in the developments of R_i, R_g, R_e.* The system of formulæ is as follows:—

$$\begin{aligned}
k^2\rho^2\,\mathrm{sn}^2\,\tfrac{1}{3}K &= k^2\rho^2\,\mathrm{cd}^2\,\tfrac{2}{3}K \\
&= -R_i+4\Sigma_1^\infty(-1)^n\Delta'(n)\,q^n-36\Sigma_1^\infty(-1)^n\Delta'(n)\,q^{3n}, \\
k^2\rho^2\,\mathrm{cn}^2\,\tfrac{1}{3}K &= k^2k'^2\rho^2\,\mathrm{sd}^2\,\tfrac{2}{3}K \\
&= R_g-4\Sigma_1^\infty(-1)^n\Delta'(n)\,q^n+36\Sigma_1^\infty(-1)^n\Delta'(n)\,q^{3n}, \\
\rho^2\,\mathrm{dn}^2\,\tfrac{1}{3}K &= k'^2\rho^2\,\mathrm{nd}^2\,\tfrac{2}{3}K \\
&= R_e-4\Sigma_1^\infty(-1)^n\Delta'(n)\,q^n+36\Sigma_1^\infty(-1)^n\Delta'(n)\,q^{3n};
\end{aligned}$$

$$\begin{aligned}
k^2\rho^2\,\mathrm{cd}^2\,\tfrac{1}{3}K &= k^2\rho^2\,\mathrm{sn}^2\,\tfrac{2}{3}K = -R_i+4\Sigma_1^\infty\Delta'(n)\,q^n-36\Sigma_1^\infty\Delta'(n)\,q^{3n}, \\
k^2k'^2\rho^2\,\mathrm{sd}^2\,\tfrac{1}{3}K &= k^2\rho^2\,\mathrm{cn}^2\,\tfrac{2}{3}K = R_g-4\Sigma_1^\infty\Delta'(n)\,q^n+36\Sigma_1^\infty\Delta'(n)\,q^{3n}, \\
k'^2\rho^2\,\mathrm{nd}^2\,\tfrac{1}{3}K &= \rho^2\,\mathrm{dn}^2\,\tfrac{2}{3}K = R_e-4\Sigma_1^\infty\Delta'(n)\,q^n+36\Sigma_1^\infty\Delta'(n)\,q^{3n};
\end{aligned}$$

$$\begin{aligned}
\rho^2\,\mathrm{ns}^2\,\tfrac{1}{3}K &= \rho^2\,\mathrm{dc}^2\,\tfrac{2}{3}K = -R_i+4-4\Sigma_1^\infty\zeta(n)\,q^{2n}+36\Sigma_1^\infty\zeta(n)\,q^{6n}, \\
\rho^2\,\mathrm{ds}^2\,\tfrac{1}{3}K &= k'^2\rho^2\,\mathrm{nc}^2\,\tfrac{2}{3}K = -R_g+4-4\Sigma_1^\infty\zeta(n)\,q^{2n}+36\Sigma_1^\infty\zeta(n)\,q^{6n}, \\
\rho^2\,\mathrm{cs}^2\,\tfrac{1}{3}K &= k'^2\rho^2\,\mathrm{sc}^2\,\tfrac{2}{3}K = -R_e+4-4\Sigma_1^\infty\zeta(n)\,q^{2n}+36\Sigma_1^\infty\zeta(n)\,q^{6n};
\end{aligned}$$

$$\begin{aligned}
\rho^2\,\mathrm{dc}^2\,\tfrac{1}{3}K &= \rho^2\,\mathrm{ns}^2\,\tfrac{2}{3}K = -R_i+\tfrac{4}{3}+4\Sigma_1^\infty\sigma(n)\,q^{2n}-36\Sigma_1^\infty\sigma(n)\,q^{6n}, \\
k'^2\rho^2\,\mathrm{nc}^2\,\tfrac{1}{3}K &= \rho^2\,\mathrm{ds}^2\,\tfrac{2}{3}K = -R_g+\tfrac{4}{3}+4\Sigma_1^\infty\sigma(n)\,q^{2n}-36\Sigma_1^\infty\sigma(n)\,q^{6n}, \\
k'^2\rho^2\,\mathrm{sc}^2\,\tfrac{1}{3}K &= \rho^2\,\mathrm{cs}^2\,\tfrac{2}{3}K = -R_e+\tfrac{4}{3}+4\Sigma_1^\infty\sigma(n)\,q^{2n}-36\Sigma_1^\infty\sigma(n)\,q^{6n}.
\end{aligned}$$

The functions $\sigma(n)$, $\Delta'(n)$, $\zeta(n)$ denote, respectively, the sum of the divisors of n, the sum of the divisors of n which have uneven conjugates, and the excess of the sum of the uneven divisors of n over the sum of the even divisors of n respectively.†

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* The symbols R_i, R_g, R_e are used to denote $\frac{4KI}{\pi^2}$, $\frac{4KG}{\pi^2}$, $\frac{4KE}{\pi^2}$ respectively, where $I = E-K$ and $G = E-k'^2K$. (*Quart. Journ.*, Vol. xx., p. 352, or *Proc. Camb. Phil. Soc.*, Vol. v., p. 191.)

† *Messenger*, Vol. xviii., p. 2.

32. By means of the formulæ

$$-R_i = 8\Sigma_1^\infty \Delta'(n)\, q^n, \quad R_g = 8\Sigma_1^\infty (-1)^{n-1}\Delta'(n)\, q^n,$$

$$R_e = 1+8\Sigma_1^\infty \zeta(n)\, q^{2n}, \quad R_i+R_g+R_e = 1-24\Sigma_1^\infty \sigma(n)\, q^{2n},$$*

we may express the twelve squared functions in terms of R_i, R_g, R_e, and of series proceeding by powers of q^3. Selecting one formula from each of the first three groups, and adding the three members of the last group, we find

$$-k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K+\tfrac{3}{2}R_g = 36\Sigma_1^\infty (-1)^{n-1}\Delta'(n)\, q^{3n},$$

$$-k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K-\tfrac{3}{2}R_i = 36\Sigma_1^\infty \Delta'(n)\, q^{3n},$$

$$\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K+\tfrac{3}{2}R_e = \tfrac{9}{2}+36\Sigma_1^\infty \zeta(n)\, q^{6n},$$

$$\rho^2 \operatorname{dc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{sc}^2 \tfrac{1}{3}K+\tfrac{3}{2}(R_i+R_g+R_e)$$
$$= \tfrac{9}{2}-108\Sigma_1^\infty \sigma(n)\, q^{6n}.$$

33. Denoting by $(R_i)_3$, $(R_g)_3$, $(R_e)_3$ the quantities into which R_i, R_g, R_e are converted by the change of q into q^3, we thus obtain the formulæ

$$k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K = \tfrac{3}{2}R_g-\tfrac{9}{2}(R_g)_3,$$

$$k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K = -\tfrac{3}{2}R_i+\tfrac{9}{2}(R_i)_3,$$

$$\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K = -\tfrac{3}{2}R_e+\tfrac{9}{2}(R_e)_3,$$

$$\rho^2 \operatorname{dc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{sc}^2 \tfrac{1}{3}K$$
$$= -\tfrac{3}{2}(R_i+R_g+R_e)+\tfrac{9}{2}\left\{(R_i)_3+(R_g)_3+(R_e)_3\right\}.$$

Values of I_3, G_3, E_3. § 34.

34. The formulæ of the preceding article give

$$3(R_i)_3 = R_i+\tfrac{2}{3}k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K,$$

$$3(R_g)_3 = R_g-\tfrac{2}{3}k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K,$$

$$3(R_e)_3 = R_e+\tfrac{2}{3}\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K;$$

whence, by substituting for $\operatorname{cd}^2 \tfrac{1}{3}K$, &c., their values from § 19,

we have

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* *Messenger*, Vol. XVIII., p. 61.

$$3(Z_I)_s = Z_I + \tfrac{2}{3}k\ \left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda} - \sqrt{1+\lambda}\right\}\rho^2,$$

$$3(Z_G)_s = Z_G - \tfrac{2}{3}kk'\left\{\sqrt{2(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu} \mp \sqrt{\mu-1}\right\}\rho^2,$$

$$3(Z_E)_s = Z_E + \tfrac{2}{3}k'\ \left\{\sqrt{2(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu} + \sqrt{1+\nu}\right\}\rho^2;$$

or, as we may write these equations,

$$3\rho_s I_s = \rho I + \tfrac{2}{3}k\ \left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda} - \sqrt{1+\lambda}\right\}\rho K,$$

$$3\rho_s G_s = \rho G - \tfrac{2}{3}kk'\left\{\sqrt{2(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu} \mp \sqrt{\mu-1}\right\}\rho K,$$

$$3\rho_s E_s = \rho E + \tfrac{2}{3}k'\ \left\{\sqrt{2(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu} + \sqrt{1+\nu}\right\}\rho K.$$

Writing for $3\rho_s$ its value from § 25, we find

$$I_s = \frac{I + \tfrac{2}{3}k\left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda} - \sqrt{1+\lambda}\right\}K}{\sqrt{2(1-w+w^2)^{\frac{1}{2}}+2-w} \pm \sqrt{1-w}},$$

$$G_s = \frac{G - \tfrac{2}{3}kk'\left\{\sqrt{2(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu} \mp \sqrt{\mu-1}\right\}K}{\sqrt{2(1-w+w^2)^{\frac{1}{2}}+2-w} \pm \sqrt{1-w}},$$

$$E_s = \frac{E + \tfrac{2}{3}k'\left\{\sqrt{2(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu} + \sqrt{1+\nu}\right\}K}{\sqrt{2(1-w+w^2)^{\frac{1}{2}}+2-w} \pm \sqrt{1-w}};$$

where

$$\lambda = \left(\frac{k'^2}{2k}\right)^{\frac{2}{3}},\quad \mu = \left(\frac{1}{2kk'}\right)^{\frac{2}{3}},\quad \nu = \left(\frac{k^2}{2k'}\right)^{\frac{2}{3}},\quad w = (2kk')^{\frac{2}{3}}.$$

Values of ρ_s^2, $k_s^2\rho_s^2$, &c. §§ 35, 36.

35. Since

$$E - I = K,\quad G - I = k^2K,\quad E - G = k'^2K,$$

we may deduce, from the first group of formulæ in the preceding section, the following expressions for ρ_s^2, $k_s^2\rho_s^2$, $k_s'^2\rho_s^2$:—

$$3\rho_s^2 = \rho^2 + \tfrac{2}{3}\rho^2(\mathrm{cs}^2\,\tfrac{1}{3}K - k'^2\,\mathrm{cd}^2\,\tfrac{1}{3}K),$$

$$3k_s^2\rho_s^2 = k^2\rho^2 - \tfrac{2}{3}k^2\rho^2(\mathrm{cd}^2\,\tfrac{1}{3}K + \mathrm{cn}^2\,\tfrac{1}{3}K),$$

$$3k_s'^2\rho_s^2 = k'^2\rho^2 + \tfrac{2}{3}\rho^2(\mathrm{cs}^2\,\tfrac{1}{3}K + k^2\,\mathrm{cn}^2\,\tfrac{1}{3}K);$$

whence, by substituting for $\mathrm{cd}^2\frac{1}{3}K$, &c., their values from § 19, we find

$$3\frac{\rho_3^2}{\rho^2} = 1+\tfrac{2}{3}\,k'\,\{\sqrt{2\,(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}\}$$

$$-\tfrac{2}{3}\,k\,\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1-\lambda}\},$$

$$3\frac{k_3^2\rho_3^2}{k^2\rho^2} = 1-\tfrac{2}{3}\frac{1}{k}\,\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1-\lambda}\}$$

$$-\tfrac{2}{3}\frac{k'}{k}\,\{\sqrt{2\,(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu}\mp\sqrt{\mu-1}\},$$

$$3\frac{k_3'^2\rho_3^2}{k'^2\rho^2} = 1+\tfrac{2}{3}\frac{1}{k'}\,\{\sqrt{2\,(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}\}$$

$$+\tfrac{2}{3}\frac{k}{k'}\,\{\sqrt{2\,(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu}\mp\sqrt{\mu-1}\}.$$

The corresponding values of $\frac{k_3\rho_3^2}{k\rho^2}$, $\frac{k_3'\rho_3^2}{k'\rho^2}$, $\frac{k_3k_3'\rho_3^2}{kk'\rho^2}$ are easily deducible from the formulæ by q-changes, as in §§ 21 and 25.

36. When $k=\frac{1}{\sqrt{2}}$, we have (§ 19)

$$\mathrm{cd}^2\tfrac{1}{3}K = \frac{3^{\frac{3}{4}}+3^{\frac{1}{4}}-2^{\frac{1}{2}}3^{\frac{1}{2}}}{2^{\frac{1}{2}}},$$

$$\mathrm{cn}^2\tfrac{1}{3}K = \frac{3^{\frac{3}{4}}-3^{\frac{1}{4}}}{2^{\frac{1}{2}}},$$

$$\mathrm{cs}^2\tfrac{1}{3}K = \frac{3^{\frac{3}{4}}+3^{\frac{1}{4}}+2^{\frac{1}{2}}3^{\frac{1}{2}}}{2^{\frac{3}{2}}};$$

whence, from the above formulæ,

$$3\frac{\rho_3^2}{\rho^2} = 1+2\,.\,3^{-\frac{1}{2}},$$

$$3\frac{k_3^2\rho_3^2}{k^2\rho^2} = 1+2\,.\,3^{-\frac{1}{2}}-2^{\frac{3}{2}}3^{-\frac{3}{4}},$$

$$3\frac{k_3'^2\rho_3^2}{k'^2\rho^2} = 1+2\,.\,3^{-\frac{1}{2}}+2^{\frac{3}{2}}3^{-\frac{3}{4}}.$$

From § 25, by putting $k=\frac{1}{\sqrt{2}}$, we find

$$3\frac{\rho_3}{\rho}=\frac{3^{\frac14}+3^{\frac34}}{2^{\frac12}},\quad 3\frac{k_3\rho_3}{k\rho}=2^{\frac12}3^{\frac14}-3^{\frac12},\quad 3\frac{k_3'\rho_3}{k'\rho}=2^{\frac12}3^{\frac14}+3^{\frac12};$$

which give

$$9\frac{\rho_3^2}{\rho^2}=3+2\,.\,3^{\frac12},\quad 9\frac{k_3^2\rho_3^2}{k^2\rho^2}=3+2\,.\,3^{\frac12}-3^{\frac32}3^{\frac34},\quad 9\frac{k_3'^2\rho_3^2}{k'^2\rho^2}=3+2\,.\,3^{\frac12}+2^{\frac32}3^{\frac34},$$

agreeing with the above results.

Values of $\mathrm{dc}^2\,\frac13K$, $\mathrm{nc}^2\,\frac13K$, $\mathrm{sc}^2\,\frac13K$. §§ 37, 38.

37. The four formulæ of § 33 show that

$$\mathrm{dc}^2\,\tfrac13K+k'^2\,\mathrm{nc}^2\,\tfrac13K+k'^2\,\mathrm{sc}^2\,\tfrac13K=k^2\,\mathrm{cd}^2\,\tfrac13K-k^2\,\mathrm{cn}^2\,\tfrac13K+\mathrm{cs}^2\,\tfrac13K.$$

If, therefore, we put

$$P=k\left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda}-\sqrt{1+\lambda}\right\}$$
$$-kk'\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac12}-2-\mu}\mp\sqrt{\mu-1}\right\}$$
$$+\;k'\left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu}+\sqrt{1+\nu}\right\},$$

we have
$$\mathrm{dc}^2\,\tfrac13K+k'^2\,\mathrm{nc}^2\,\tfrac13K+k'^2\,\mathrm{sc}^2\,\tfrac13K=P.$$

Now
$$\mathrm{dc}^2\,\tfrac13K+k'^2\,\mathrm{nc}^2\,\tfrac13K+k'^2\,\mathrm{sc}^2\,\tfrac13K=3\mathrm{dc}^2\,\tfrac13K-1-k^2$$
$$=3k'^2\,\mathrm{nc}^2\,\tfrac13K+k^2-k'^2$$
$$=3k'^2\,\mathrm{sc}^2\,\tfrac13K+1+k'^2;$$

and we thus obtain the curious formulæ

$$\mathrm{dc}^2\,\tfrac13K=\tfrac13\,(1+k^2+P),$$
$$k'^2\,\mathrm{nc}^2\,\tfrac13K=\tfrac13\,(k'^2-k^2+P),$$
$$k'^2\,\mathrm{sc}^2\,\tfrac13K=\tfrac13\,(-1-k'^2+P).$$

38. When $k=\frac{1}{\sqrt2}$, $P=\frac{3^{\frac32}+3^{\frac34}}{2^{\frac32}}$,

and the formulæ in the preceding section give

$$\mathrm{dc}^2\,\tfrac13K=\tfrac12+\frac{3^{\frac12}+3^{-\frac14}}{2^{\frac32}},$$

$$\operatorname{nc}^2 \tfrac13 K = \frac{3^{\frac12}+3^{-\frac12}}{2^{\frac12}},$$

$$\operatorname{sc}^2 \tfrac13 K = -\tfrac12 + \frac{3^{\frac12}+3^{-\frac12}}{2^{\frac12}};$$

which are easily seen to be equal to the reciprocals of the values of $\operatorname{cd}^2 \frac13 K$, &c., given in § 36.

Elliptic and Zeta Functions of $\frac12 K$ and $\frac14 K$. §§ 39–45.

39. Of the q-series for the sixteen elliptic and zeta functions, eight proceed by uneven multiples of the argument x, and eight by even multiples. By putting $x = \frac14\pi$ in the former, we obtain the following formulæ depending upon a new function $T(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$ or 3, mod. 8, over the number of divisors which $\equiv 5$ or 7, mod. 8:—

$$k\rho \operatorname{sn} \tfrac12 K = \quad k\rho \operatorname{cd} \tfrac12 K = 2\sqrt{2}\, \Sigma_1^\infty\, T(m)\, q^{\frac12 m},$$

$$k\rho \operatorname{cn} \tfrac12 K = kk'\rho \operatorname{sd} \tfrac12 K = 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^{\frac12(m-1)}\, T(m)\, q^{\frac12 m},$$

$$\rho \operatorname{ns} \tfrac12 K = \quad \rho \operatorname{dc} \tfrac12 K = \sqrt{2} + 2\sqrt{2}\, \Sigma_1^\infty\, T(n)\, q^n,$$

$$\rho \operatorname{ds} \tfrac12 K = \; k'\rho \operatorname{nc} \tfrac12 K = \sqrt{2} + 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^n\, T(n)\, q^n.$$

Substituting for $\operatorname{sn} \frac12 K$, &c., their values, we thus find

$$(1-k')^{\frac12}\rho = 2\sqrt{2}\, \Sigma_1^\infty\, T(m)\, q^{\frac12 m},$$

$$k'^{\frac12}(1-k')^{\frac12}\rho = 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^{\frac12(m-1)}\, T(m)\, q^{\frac12 m},$$

$$(1+k')^{\frac12}\rho = \sqrt{2} + 2\sqrt{2}\, \Sigma_1^\infty\, T(n)\, q^n,$$

$$k'^{\frac12}(1+k')^{\frac12}\rho = \sqrt{2} + 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^n\, T(n)\, q^n.$$

40. We do not obtain new results by putting $x = \frac14\pi$ in the eight formulæ proceeding by even powers of x, the equations so obtained being

$$\tfrac12 (1-k')\,\rho = 4\Sigma_1^\infty\, E(m)\, q^m,$$

$$\tfrac12 (1+k')\,\rho = 1 + 4\Sigma_1^\infty\, E(n)\, q^{2n},$$

$$k'^{\frac12}\rho = 1 + 4\Sigma_1^\infty\, (-1)^n\, E(n)\, q^{2n},$$

which are deducible at once from the q-series for $k\rho$, ρ, and $k'\rho$, by changing q into q^2.

41. We may, however, obtain results involving the argument $\frac14$K by putting $x = \frac18\pi$ *in the eight* formulæ proceeding by even multiples

of x. The q-series which occur in these formulæ do not introduce any new arithmetical function, being expressible by means of the functions T and E, as follows:—

$$\rho \operatorname{zn} \tfrac{1}{4}K = -\rho \operatorname{zd} \tfrac{3}{4}K = 2\sqrt{2}\, \Sigma_1^\infty\, T(m)\, q^m + 4\Sigma_1^\infty\, E(m)\, q^{2m},$$

$$\rho \operatorname{zd} \tfrac{1}{4}K = -\rho \operatorname{zn} \tfrac{3}{4}K = -2\sqrt{2}\, \Sigma_1^\infty\, T(m)\, q^m + 4\Sigma_1^\infty\, E(m)\, q^{2m};$$

$$\rho \operatorname{dn} \tfrac{1}{4}K = \quad k'\rho \operatorname{nd} \tfrac{3}{4}K$$

$$= 1 + 2\sqrt{2}\Sigma_1^\infty\, (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^m + 4\Sigma_1^\infty\, (-1)^n\, E(n)\, q^{4n},$$

$$k'\rho \operatorname{nd} \tfrac{1}{4}K = \quad \rho \operatorname{dn} \tfrac{3}{4}K$$

$$= 1 - 2\sqrt{2}\Sigma_1^\infty\, (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^m + 4\Sigma_1^\infty\, (-1)^n\, E(n)\, q^{4n};$$

$$\rho \operatorname{zs} \tfrac{1}{4}K = -\rho \operatorname{zc} \tfrac{3}{4}K = \quad \cot \tfrac{1}{8}\pi + 2\sqrt{2}\, \Sigma_1^\infty\, T(n)\, q^{2n} + 4\Sigma_1^\infty\, E(n)\, q^{4n},$$

$$\rho \operatorname{cs} \tfrac{1}{4}K = \quad k\rho \operatorname{sc} \tfrac{3}{4}K$$

$$= \quad \cot \tfrac{1}{8}\pi + 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^n\, T(n) q^{2n} + 4\Sigma_1^\infty\, (-1)^n\, E(n)\, q^{4n};$$

$$\rho \operatorname{zc} \tfrac{1}{4}K = -\rho \operatorname{zs} \tfrac{3}{4}K = -\tan \tfrac{1}{8}\pi - 2\sqrt{2}\, \Sigma_1^\infty\, T(n)\, q^{2n} + 4\Sigma_1^\infty\, E(n)\, q^{4n},$$

$$k'\rho \operatorname{sc} \tfrac{1}{4}K = \quad \rho \operatorname{cs} \tfrac{3}{4}K$$

$$= \quad \tan \tfrac{1}{8}\pi + 2\sqrt{2}\, \Sigma_1^\infty\, (-1)^n\, T(n)\, q^{2n} - 4\Sigma_1^\infty (-1)^n\, E(n)\, q^{4n}.$$

42. By changing q into q^2 in the T-series at the end of § 39, and the E-series in § 40, we find

$$2\sqrt{2}\,\Sigma_1^\infty\, T(m)\, q^m = \tfrac{1}{2}(1-k'^{\frac{1}{2}})(1+k')^{\frac{1}{2}}\rho,$$

$$2\sqrt{2}\,\Sigma_1^\infty\, (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^m = 2^{-\frac{1}{2}} k^{\frac{1}{2}}\, (1-k'^{\frac{1}{2}})\, \rho,$$

$$\sqrt{2} + 2\sqrt{2}\,\Sigma_1^\infty\, T(n)\, q^{2n} = \tfrac{1}{2}(1+k'^{\frac{1}{2}})(1+k')^{\frac{1}{2}}\rho,$$

$$\sqrt{2} + 2\sqrt{2}\,\Sigma_1^\infty\, (-1)^n\, T(n)\, q^{2n} = 2^{-\frac{1}{2}} k^{\frac{1}{2}}\, (1+k'^{\frac{1}{2}})\, \rho;$$

and

$$4\Sigma_1^\infty\, E(m)\, q^{2m} = \tfrac{1}{4}(1-k'^{\frac{1}{2}})^2\rho,$$

$$1 + 4\Sigma_1^\infty\, E(n)\, q^{4n} = \tfrac{1}{4}(1+k'^{\frac{1}{2}})^2\rho,$$

$$1 + 4\Sigma_1^\infty\, (-1)^n\, E(n)\, q^{4n} = 2^{-\frac{1}{2}} k^{\frac{1}{2}}(1+k')^{\frac{1}{2}}\rho.$$

43. Substituting these values for the q-series in § 41, and separating into different groups the zeta functions and the elliptic functions, we find

$$\operatorname{zs} \tfrac{1}{4}K = \tfrac{1}{4}(1+k'^{\frac{1}{2}})\left\{1+k'^{\frac{1}{2}}+2(1+k')^{\frac{1}{2}}\right\},$$

$$\operatorname{zc} \tfrac{1}{4}K = \tfrac{1}{4}(1+k'^{\frac{1}{2}})\left\{1+k'^{\frac{1}{2}}-2(1+k')^{\frac{1}{2}}\right\},$$

$$\operatorname{zd} \tfrac{1}{4}K = \tfrac{1}{4}(1-k'^{\frac{1}{2}})\left\{1-k'^{\frac{1}{2}}-2(1+k')^{\frac{1}{2}}\right\},$$

$$zn\, \tfrac{1}{4}K = \tfrac{1}{4}(1-k'^{\frac{1}{2}})\left\{1-k'^{\frac{1}{2}}+2(1+k')^{\frac{1}{2}}\right\};$$

and

$$\operatorname{cs}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\{1+k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\},$$

$$k'\operatorname{sc}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\{1+k'^{\frac{1}{2}}-(1+k')^{\frac{1}{2}}\},$$

$$\operatorname{dn}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\{1-k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\},$$

$$k'\operatorname{nd}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\{-1+k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\}.$$

44. Putting for the moment $p^{\frac{1}{2}}$ in place of $(1+k')^{\frac{1}{2}}$, the zeta formulæ may be written

$$\operatorname{zs}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'+2k'^{\frac{1}{2}}+2p^{\frac{1}{2}}+2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$

$$\operatorname{zc}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'+2k'^{\frac{1}{2}}-2p^{\frac{1}{2}}-2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$

$$\operatorname{zd}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'-2k'^{\frac{1}{2}}-2p^{\frac{1}{2}}+2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$

$$\operatorname{zn}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'-2k'^{\frac{1}{2}}+2p^{\frac{1}{2}}-2k'^{\frac{1}{2}}p^{\frac{1}{2}});$$

whence it is evident that

$$\operatorname{zs}\tfrac{1}{4}K-\operatorname{zn}\tfrac{1}{4}K = k'^{\frac{1}{2}}+k'^{\frac{1}{2}}p^{\frac{1}{2}} = \operatorname{cs}\tfrac{1}{4}K\operatorname{dn}\tfrac{1}{4}K,$$

$$\operatorname{zn}\tfrac{1}{4}K-\operatorname{zc}\tfrac{1}{4}K = -k'^{\frac{1}{2}}+p^{\frac{1}{2}} = \operatorname{sc}\tfrac{1}{4}K\operatorname{dn}\tfrac{1}{4}K.$$

The functions thus satisfy the equations

$$\operatorname{zs}x-\operatorname{zn}x = \operatorname{cs}x\operatorname{dn}x,\quad \operatorname{zn}x-\operatorname{zc}x = \operatorname{sc}x\operatorname{dn}x,$$

as they should do.

45. It can be shown that

$$\operatorname{zs}2x = \tfrac{1}{2}(\operatorname{zs}x+\operatorname{zc}x+\operatorname{zd}x+\operatorname{zn}x),$$

$$\operatorname{ns}2x = \tfrac{1}{2}(\operatorname{zs}x-\operatorname{zc}x-\operatorname{zd}x+\operatorname{zn}x),$$

$$\operatorname{ds}2x = \tfrac{1}{2}(\operatorname{zs}x-\operatorname{zc}x+\operatorname{zd}x-\operatorname{zn}x),$$

$$\operatorname{cs}2x = \tfrac{1}{2}(\operatorname{zs}x+\operatorname{zc}x-\operatorname{zd}x-\operatorname{zn}x),$$

and these formulæ afford another verification of the values of the zeta functions in § 44; for, putting $x = \frac{1}{4}\pi$, and substituting for $\operatorname{zs}x$, &c., we find

$$\operatorname{zs}\tfrac{1}{2}K = \tfrac{1}{2}(1+k'),\quad \operatorname{ns}\tfrac{1}{2}K = p^{\frac{1}{2}},\quad \operatorname{ds}\tfrac{1}{2}K = k'^{\frac{1}{2}}p^{\frac{1}{2}},\quad \operatorname{cs}\tfrac{1}{2}K = k'^{\frac{1}{2}},$$

which are the correct values of these quantities. The formulæ in this section evidently afford a very simple means of obtaining the values of $\operatorname{zs}\frac{1}{4}K$, &c. in the first instance.

Representation of Numbers by the Form x^2+2y^2. § 46.

46. Since $\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{n^2}, \quad k^{\frac12}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{\frac14 m^2},$

we have, by changing q into q^2,

$$\frac{(1+k')^{\frac12}}{2^{\frac12}}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{2n^2}, \quad \frac{(1-k')^{\frac12}}{2^{\frac12}}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{\frac12 m^2},$$

so that
$$(1+k')^{\frac12}\rho = \sqrt2\, \Sigma_{-\infty}^{\infty} q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{2n^2},$$
$$(1-k')^{\frac12}\rho = \sqrt2\, \Sigma_{-\infty}^{\infty} q^{\frac12 m^2} \times \Sigma_{-\infty}^{\infty} q^{n^2}.$$

Comparing these formulæ with the second group in § 39, we see that

$$\Sigma_{-\infty}^{\infty} q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{2n^2} = 1 + 2\Sigma_1^{\infty}\, T(n)\, q^n,$$
$$\Sigma_{-\infty}^{\infty} q^{\frac12 m^2} \times \Sigma_{-\infty}^{\infty} q^{n^2} = 2\Sigma_1^{\infty}\, T(m)\, q^{\frac12 m}.$$

The first of these formulæ shows that the number of representations of a number by the form x^2+2y^2 is equal to $2T(n)$.* The second formula may be written

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{2m^2} = 2\Sigma_1^{\infty}\, T(m)\, q^m,$$

and, as regards arithmetical interpretation, is included in the first, from which it is easily deducible.

The Function $T(n)$. § 47.

47. The function $T(n)$, like so many other functions of the same class,† possesses the property typified by

$$\phi(pq) = \phi(p)\,\phi(q),$$

p and q being relatively prime, and also satisfies a recurring relation in which the arguments differ from the highest argument by squared numbers.

It is easily seen that, if a be a prime $\equiv 1$ or 3, mod. 8, then

$$T(a^{\alpha}) = \alpha+1;$$

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* This theorem is due to Lejeune-Dirichlet (*Crelle's Journal*, Vol. xxi., p. 3).

† *See Proc. Lond. Math. Soc.*, Vol. xxi., p. 214 (§ 56).

and that, if r be a prime $\equiv 5$ or 7, mod. 8, then

$$T(r^\rho) = 1 \text{ or } 0,$$

according as ρ is even or uneven.

In general, if $$n = 2^p a^\alpha b^\beta c^\gamma \ldots r^\rho s^\sigma t^\tau \ldots,$$

where $a, b, c, \ldots$ are primes which $\equiv 1$ or 3, mod. 8, and $r, s, t, \ldots$ are primes which $\equiv 5$ or 7, mod. 8, then $T(n)$ is zero, unless $\rho, \sigma, \tau, \ldots$ are all even, and, if they are all even, then

$$T(n) = (\alpha+1)(\beta+1)(\gamma+1) \ldots;$$

so that, if $$n = n_1 n_2 n_3 \ldots,$$

where $n_1, n_2, n_3 \ldots$ are relatively prime, then

$$T(n) = T(n_1)\, T(n_2)\, T(n_3) \ldots.$$

It is easy to see that, if $n \equiv 5$ or 7, mod. 8, $T(n)$ vanishes; for to every divisor which $\equiv 1$ or 3, mod. 8, there corresponds one which $\equiv 5$ or 7, mod. 8, and *vice versâ*.*

As regards all its principal properties, therefore, the function $T(n)$ is closely analogous to $E(n)$ and $H(n)$.

48. From § 39, by division, we find

$$k'^{\frac{1}{2}} = \frac{1+2\Sigma_1^\infty (-1)^n\, T(n)\, q^n}{1+2\Sigma_1^\infty T(n)\, q^n} = \frac{\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^{\frac{1}{2}m}}{\Sigma_1^\infty T(m)\, q^{\frac{1}{2}m}},$$

and we know that $$k'^{\frac{1}{2}} = \frac{\Sigma_{-\infty}^\infty (-1)^n q^n}{\Sigma_{-\infty}^\infty q^{n^2}}.$$

We thus obtain the identical formulæ

$$\frac{\Sigma_{-\infty}^\infty (-1)^n q^{n^2}}{\Sigma_{-\infty}^\infty q^{n^2}} = \frac{1+2\Sigma_1^\infty (-1)^n\, T(n)\, q^n}{1+2\Sigma_1^\infty T(n)\, q^n},$$

$$\frac{\Sigma_{-\infty}^\infty (-1)^n q^{2n^2}}{\Sigma_{-\infty}^\infty q^{2n^2}} = \frac{\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^m}{\Sigma_1^\infty T(m)\, q^m};$$

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* It follows, therefore, from § 42, that a number $\equiv 5$ or 7 cannot be represented by the form x^2+2y^2. This is however obvious; for, by considering separately the four cases of x and y even or uneven, we see that x^2+2y^2 must be either (i.) even, or (ii.) uneven and $\equiv 1$ or 3, mod. 8.

whence, by equating coefficients, we find that, m being any uneven number,

$$T(m)-2T(m-1)+2T(m-4)-2T(m-9)+\ldots=0,^{*}$$

and that, r being any number which $\equiv 3$, mod. 4,

$$T(r)-2T(r-2)+2T(r-8)-2T(r-18)+\ldots=0.$$

In the first formula, $T(0)$, when it occurs, is to have the value $\tfrac{1}{2}$.

In the second formula the numbers 2, 8, 18, ... are the double of the squares. If $r\equiv 7$, mod. 8, all the terms vanish, as all the arguments are $\equiv 5$ or 7, mod. 8. We may therefore suppose $r\equiv 3$, mod. 8, without substantial loss of generality.

Developments of $k^2\rho^2$ sn² ¼K, &c., in ascending powers of q. § 49.

49. Proceeding as in § 31, we may deduce from the q-series for the squared elliptic functions the expansions of $k^2\rho^2\,\mathrm{sn}^2\,\tfrac{1}{4}K$, &c., in ascending powers of q. These developments involve $\Delta'(n)$ and $\zeta(n)$, and also a new arithmetical function $V(n)$, defined as denoting the excess of the sum of the divisors of n which $\equiv 1$ or 7, mod. 8, over the sum of the divisors which $\equiv 3$ or 5, mod. 8.

Selecting one formula from each group, the developments are

$$\begin{aligned}k^2\rho^2\,\mathrm{cn}^2\,\tfrac{1}{4}K-R_g&=k^2k'^2\rho^2\,\mathrm{sd}^2\,\tfrac{3}{4}K-R_g\\&=4\sqrt{2}\,\Sigma_1^\infty\,V(m)\,q^m+32\,\Sigma_1^\infty\,(-1)^n\,\Delta'(n)\,q^{4n},\end{aligned}$$

$$\begin{aligned}-k^2k'^2\rho^2\,\mathrm{sd}^2\,\tfrac{1}{4}K+R_g&=-k^2\rho^2\,\mathrm{cn}^2\,\tfrac{3}{4}K+R_g\\&=4\sqrt{2}\,\Sigma_1^\infty\,V(m)\,q^m-32\,\Sigma_1^\infty\,(-1)^n\,\Delta'(n)\,q^{4n},\end{aligned}$$

$$\begin{aligned}-\rho^2\,\mathrm{ds}^2\,\tfrac{1}{4}K-R_g+\mathrm{cec}^2\,\tfrac{1}{8}\pi&=-k'^2\rho^2\,\mathrm{nc}^2\,\tfrac{3}{4}K+R_g-\sec^2\,\tfrac{3}{8}\pi\\&=4\sqrt{2}\,\Sigma_1^\infty\,V(n)\,q^{2n}-32\,\Sigma_1^\infty\,\zeta(n)\,q^{8n},\end{aligned}$$

$$\begin{aligned}{}'^2\rho^2\,\mathrm{nc}^2\,\tfrac{1}{4}K+R_g-\sec^2\,\tfrac{1}{8}\pi&=-\rho^2\,\mathrm{ds}^2\,\tfrac{3}{4}K+R_g-\mathrm{cec}^2\,\tfrac{3}{8}\pi\\&=4\sqrt{2}\,\Sigma_1^\infty\,V(n)\,q^{2n}+32\,\Sigma_1^\infty\,\zeta(n)\,q^{8n}.\end{aligned}$$

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* Similar recurring formulæ relating to other arithmetical functions are given in *Quart. Journ.*, Vol. xx., p. 121, and *Proc. Lond. Math. Soc.*, Vol. xxi., pp. 205, 210.

Values of q-series whose coefficients are expressed by the function V.

§§ 50–54.

50. It can be shown that, by the change of q into q^4, R_g and R_e are converted into

$$\tfrac{1}{4}R_e-\tfrac{1}{4}k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2 \quad\text{and}\quad \tfrac{1}{4}R_e+\tfrac{1}{4}k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2,$$

respectively. Thus, from the formulæ (§ 32)

$$8\Sigma_1^\infty(-1)^{n-1}\Delta'(n)\,q^n=R_g,\quad 1+8\Sigma_1^\infty\zeta(n)\,q^{2n}=R_e$$

we may deduce that

$$32\Sigma_1^\infty(-1)^n\,\Delta'(n)\,q^{4n}=R_e-k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$
$$4+32\Sigma_1^\infty\zeta(n)\,q^{8n}=R_e+k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2.$$

51. Substituting these values in the first and third equations of § 49, we find

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m=k^2\rho^2\,\mathrm{cn}^2\,\tfrac{1}{4}K-R_g+R_e-k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$
$$4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n}=-\rho^2\,\mathrm{ds}^2\,\tfrac{1}{4}K-R_g+\mathrm{cec}^2\,\tfrac{1}{8}\pi-4+R_e+k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\rho^2,$$

giving

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m=\rho^2\,\mathrm{dn}^2\,\tfrac{1}{4}K-k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$
$$4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n}=-\rho^2\,\mathrm{cs}^2\,\tfrac{1}{4}K+2\sqrt{2}+k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2.$$

Putting for dn $\frac{1}{4}K$ and cs $\frac{1}{4}K$ their values from § 43, we find ultimately

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m=k'^{\frac{1}{2}}(1+k')^{\frac{1}{2}}(1-k'^{\frac{1}{2}})\,\rho^2,$$
$$2\sqrt{2}-4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n}=k'^{\frac{1}{2}}(1+k')^{\frac{1}{2}}(1+k'^{\frac{1}{2}})\,\rho^2.$$

The second and fourth equations of § 49 lead also to the same formulæ.

52. By changing q into $q^{\frac{1}{2}}$ in these formulæ, we find

$$4\,\Sigma_1^\infty V(m)\,q^{\frac{1}{2}m}=k'\left\{(1+k)^{\frac{1}{2}}-(1-k)^{\frac{1}{2}}\right\}\rho^2,$$
$$2-4\,\Sigma_1^\infty V(n)\;q^n\;=k'\left\{(1+k)^{\frac{1}{2}}+(1-k)^{\frac{1}{2}}\right\}\rho^2.$$

These equations may also be written in the more convenient forms

$$2\sqrt{2}\,\Sigma_1^\infty V(m)\,q^{\frac{1}{2}m}=k'(1-k')^{\frac{1}{2}}\,\rho^2,$$
$$\sqrt{2}-2\sqrt{2}\,\Sigma_1^\infty V(n)\;q^n\;=k'(1+k')^{\frac{1}{2}}\,\rho^2.$$

Changing the sign of q, we have also

$$2\sqrt{2}\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,V(m)\,q^{\frac12 m} = k'^{\frac12}\,(1-k')^{\frac12}\,\rho^2,$$

$$\sqrt{2}-2\sqrt{2}\,\Sigma_1^\infty\,(-1)^n\,V(n)\,q^n = k'^{\frac12}\,(1+k')^{\frac12}\,\rho^2.$$

There is a remarkable resemblance between these equations and the corresponding results involving the function T (§ 39).

53. By changing q into $q^{\frac12}$ in the four preceding formulæ, and also in the corresponding group of T-formulæ (§ 39), we find

$$2\,\Sigma_1^\infty\,T(m)\,q^{\frac14 m} = k^{\frac12}\,(1+k)^{\frac12}\,\rho,$$

$$2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,T(m)\,q^{\frac14 m} = k^{\frac12}\,(1-k)^{\frac12}\,\rho,$$

$$1+2\,\Sigma_1^\infty\,T(n)\,q^{\frac12 n} = (1+k)^{\frac12}\,\rho,$$

$$1+2\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^{\frac12 n} = (1-k)^{\frac12}\,\rho,$$

$$2\,\Sigma_1^\infty\,V(m)\,q^{\frac14 m} = k'k^{\frac12}\,(1-k)^{\frac12}\,\rho^2,$$

$$2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,V(m)\,q^{\frac14 m} = k'k^{\frac12}\,(1+k)^{\frac12}\,\rho^2,$$

$$1-2\,\Sigma_1^\infty\,V(n)\,q^{\frac12 n} = k'\,(1-k)^{\frac12}\,\rho^2,$$

$$1-2\,\Sigma_1^\infty\,(-1)^n\,V(n)\,q^{\frac12 n} = k'\,(1+k)^{\frac12}\,\rho^2.$$

54. It may be remarked that

$$1+24\,\Sigma_1^\infty\,\Delta(n)\,q^n = (1+k^2)\,\rho^2, \qquad 4\Sigma_1^\infty\,\Delta(m)\,q^{\frac12 m} = k\rho^2,\text{*}$$

where $\Delta(n)$ denotes the sum of the uneven divisors of n; so that, by combining these formulæ with

$$\sqrt{2}-2\sqrt{2}\,\Sigma_1^\infty\,V(n)\,q^n = k'\,(1+k')^{\frac12}\,\rho^2,\quad 2\sqrt{2}\,\Sigma_1^\infty\,V(m)\,q^{\frac12 m} = k'(1-k')^{\frac12}\rho^2,$$

we obtain at once the values of the series

$$\Sigma_1^\infty\,\mathrm{v}(n)\,q^n,\quad \Sigma_1^\infty\,\mathrm{u}(n)\,q^n,\quad \Sigma_1^\infty\,\mathrm{v}(m)\,q^m,\quad \Sigma_1^\infty\,\mathrm{u}(m)\,q^m,$$

where $\mathrm{v}(n)$ denotes the sum of the divisors of n which $\equiv 1$ or 7, mod. 8, and $\mathrm{u}(n)$ denotes the sum of the divisors of n which $\equiv 5$ or 7, mod. 8.

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* *Messenger of Mathematics*, Vol. XVIII., p. 61.

The function $V(n)$. §§ 55, 56.

55. By general reasoning of the kind explained in the note on p. 186 of Vol. XXI., we can see that, if p and q are relatively prime,

$$V(pq) = V(p)\,V(q).$$

It is, however, easy to obtain an expression for $V(n)$ in terms of the prime divisors of n, which affords also an analytical proof of this theorem.

For, if $n = a^{\alpha}$, where a is a prime which $\equiv$ 1 or 7, mod. 8, then

$$V(n) = \frac{a^{\alpha+1}-1}{a-1};$$

if $n = r^{\rho}$, where r is a prime which $\equiv$ 3 or 5, mod. 8, then

$$V(n) = (-1)^{\rho}\frac{r^{\rho+1}+(-1)^{\rho}}{r+1},$$

and, in general, if $\qquad n = a^{\alpha} b^{\beta} c^{\gamma} \dots r^{\rho} s^{\sigma} t^{\tau} \dots,$

where $a, b, c, \dots$ are primes which $\equiv$ 1 or 7, mod. 8, and $r, s, t, \dots$ are primes which $\equiv$ 3 or 5, mod. 8, then

$$V(n) = (-1)^{\rho+\sigma+\tau+\dots}\frac{a^{\alpha+1}-1}{a-1}\,\frac{b^{\beta+1}-1}{b-1}\,\frac{c^{\gamma+1}-1}{c-1}\dots$$

$$\dots\frac{r^{\rho+1}+(-1)^{\rho}}{r+1}\,\frac{s^{\sigma+1}+(-1)^{\sigma}}{s+1}\,\frac{t^{\tau+1}+(-1)^{\tau}}{t+1}\dots.$$

56. From § 52, we deduce

$$k'^{\frac{1}{2}} = \frac{1-2\,\Sigma_1^{\infty}\,V(n)\,q^{n}}{1-2\,\Sigma_1^{\infty}{}'\,(-1)^{n}\,V(n)\,q^{n}} = \frac{\Sigma_1^{\infty}\,V(m)\,q^{\frac{1}{2}m}}{\Sigma_1^{\infty}\,(-1)^{\frac{1}{2}(m-1)}\,V(m)\,q^{\frac{1}{2}m}};$$

whence, by following the same process as in § 48, we obtain the recurring formulæ

$$V(m)+2V(m-1)+2V(m-4)+2V(m-9)+\dots = 0,$$

$$V(r)+2V(r-2)+2V(r-8)+2V(r-18)+\dots = 0,$$

where, as in § 48, m is any uneven number, and r is any number $\equiv$ 3, mod. 4. The quantity $V(0)$, when it occurs, is to have the value $-\frac{1}{2}$.

Thursday, January 8th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

The following gentlemen were elected members: — H. Taber, Docent in Clark University, Massachusetts; James Buchanan, B.A., late Scholar of Peterhouse, Cambridge; J. L. S. Hatton, B.A., Scholar of Hertford College; and A. W. Flux, B.A., Fellow of St. John's College, Cambridge.

The following communications were made:—

Geometrical Metamorphoses by Partition and Transformation: H. Perigal.

The Theory of Perfect Partitions and the Compositions of Multipartite Numbers: Major Macmahon.

On a Certain Class of Plane Quartics: Prof. G. B. Mathews.

The following presents were received:—

"Educational Times," for January.

"Journal of the Institute of Actuaries," Vol. xix., Parts 2–6; Vol. xx., Parts 1 and 5; Vol. xxii., Parts 2 and 6; Vol. xxiii., Parts 1, 2, 4, 5, 6; Vol. xxiv., Parts 2–6; Vol. xxv., Parts 1, 2, 4, 5, 6; Vol. xxvi., Parts 1–5; Vol. xxvii., Parts 1, 3, 4, 5, 6; Vol. xxviii.

"Bulletin des Sciences Mathématiques," Dec., 1890.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 119, 120.

"Beiblätter zu den Annalen der Physik und Chemie," Band xiv., Stück 11.

"Annales de l'École Polytechnique de Delft," Tome v., 1re Livraison; Leyde, 1890.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Math.-phys. Classe, 1890, ii.; Leipzig.

"Rendiconti del Circolo Matematico di Palermo," Tomo iv., Fasc. vi.; Nov.–Dic., 1890.

"Jahrbuch über die Fortschritte der Mathematik," Band xx., Heft 1; Jahrgang, 1888; Berlin, 1890.

"Acta Mathematica," xiii., 3 and 4.

"Annali di Matematica," Tomo xviii., Fasc. 4.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vi., Fasc. 8 and 9; Roma.

"Nyt Tidsskrift for Mathematik," A. Første Aargang, Nos. 4, 5, 6, 7; B. Første Aargang, No. 3; Copenhagen.

On a Certain Class of Plane Quartics. *By* Prof. G. B. Mathews.

[*Read Jan.* 8*th*, 1891.]

A quartic curve of the most general kind may always be obtained as the envelope of the first polars of points on a conic with respect to a fixed cubic. The determination of the cubic involves nine conditions, and that of the conic five more; in general, therefore, when the quartic is given, the problem of finding a suitable conic and cubic admits of a finite number of solutions; the number is, in fact, sixty-three. Suppose, however, that the S invariant of the cubic vanishes; then the canonical equation of the cubic may be taken to be

$$x^3+y^3+z^3=0 \text{ ....} (1);$$

and if

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \text{ ....} (2)$$

is the conic, then the quartic is

$$\begin{vmatrix} a & h & g & x^2 \\ h & b & f & y^2 \\ g & f & c & z^2 \\ x^2 & y^2 & z^2 & 0 \end{vmatrix} = 0,$$

or

$$\phi \equiv Ax^4+By^4+Cz^4+2Fy^2z^2+2Gz^2x^2+2Hx^2y^2=0 \text{ ....} (3).$$

Now, if λ, μ, ν are any three constants, none of which is zero, the equation of the quartic may be expressed in the form

$$\begin{vmatrix} a\lambda^2 & h\lambda\mu & g\nu\lambda & \lambda x^2 \\ h\lambda\mu & b\mu^2 & f\mu\nu & \mu y^2 \\ g\nu\lambda & f\mu\nu & c\nu^2 & \nu z^2 \\ \lambda x^2 & \mu y^2 & \nu z^2 & 0 \end{vmatrix} = 0,$$

and hence there is a doubly infinite system of conics and cubics from which the quartic may be derived, namely,

$$a\lambda^2x^2+b\mu^2y^2+c\nu^2z^2+2f\mu\nu yz+2g\nu\lambda zx+2h\lambda\mu xy=0 \text{ ....} (4),$$

$$\lambda x^3+\mu y^3+\nu z^3=0 \text{ ....} (5).$$

All the cubics (5) have the same Hessian,

$$xyz = 0.$$

The line $x = 0$ cuts (4) in two points, given by

$$b\mu^2y_1^2+2f\mu\nu y_1z_1+c\nu^2z_1^2 = 0,$$

and, combining this with $\mu y_1y^2+\nu z_1z^2 = 0$,

we get $$bz^4-2fy^2z^2+cy^4 = 0 \quad\text{........................}(6)$$

for the equation of the two pairs of double tangents to the quartic which go through the point ($y = 0$, $z = 0$).

Similarly, $$cx^4-2gz^2x^2+az^4 = 0,$$

$$ay^4-2hx^2y^2+bz^4 = 0$$

represent four other pairs of double tangents.

Observing that

$$A\phi = (Ax^2+Hy^2+Gz^2)^2+\Delta\,(cy^4-2fy^2z^2+bz^4),$$

where $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

we see that the eight points of contact of the four tangents

$$cy^4-2fy^2z^2+bz^4 = 0,$$

lie on the conic $$Ax^2+Hy^2+Gz^2 = 0 \quad\text{........................}(7).$$

Similarly, $$Hx^2+By^2+Fz^2 = 0,$$

$$Gx^2+Fy^2+Cz^2 = 0,$$

pass each through eight points of contact of four double tangents. In order to find the other double tangents, put

$$cy^4-2fy^2z^2+bz^4 = c\,(y^2-m_1^2z^2)(y^2-m_2^2z^2) \quad\text{...........}(8),$$

then the conic

$$\lambda^2\,(y-m_1z)(y-m_2z)+2\lambda\,(Ax^2+Hy^2+Gz^2)$$
$$-\Delta c\,(y+m_1z)(y+m_2z) = 0 \quad\text{...........}(9)$$

touches the quartic in four points, and will represent a pair of double tangents if

$$\begin{vmatrix} 2\lambda A & 0 & 0 \\ 0 & \lambda^2+2H\lambda-\Delta c & \tfrac{1}{2}(\lambda^2+\Delta c)(m_1+m_2) \\ 0 & \tfrac{1}{2}(\lambda^2+\Delta c)(m_1+m_2) & (\lambda^2-\Delta c)\,m_1m_2+2\lambda G \end{vmatrix} = 0.$$

The roots $\lambda = 0$ and $\lambda = \infty$ give

$$(y+m_1z)(y+m_2z) = 0 \quad \text{and} \quad (y-m_1z)(y-m_2z) = 0.$$

The other roots satisfy the equation

$$(m_1-m_2)^2\lambda^4-8(G+Hm_1m_2)\lambda^3+\left\{2\Delta c(m_1+m_2)^2+8\Delta c m_1m_2-16GH\right\}\lambda^2$$
$$+8\Delta c\,(G+Hm_1m_2)\,\lambda+(m_1-m_2)^2\,\Delta^2c^2 = 0.$$

Now
$$m_1^2+m_2^2 = \frac{2f}{c},$$
$$m_1^2m_2^2 = \frac{b}{c};$$

therefore
$$(m_1-m_2)^2 = \frac{2\,(f-\sqrt{bc})}{c},$$
$$G+Hm_1m_2 = \frac{(hf-bg)\,c+(fg-ch)\sqrt{bc}}{c}$$
$$= \frac{(f-\sqrt{bc})(ch+g\sqrt{bc})}{c};$$

and
$$2\Delta c\,(m_1+m_2)^2+8\Delta c m_1m_2-16GH$$
$$= 4\Delta f+12\Delta\sqrt{bc}-16GH$$
$$= -12\Delta\,(f-\sqrt{bc})+16\,(\Delta f-GH)$$
$$= -12\Delta\,(f-\sqrt{bc})-16AF$$
$$= 4\,(f-\sqrt{bc})\,\left\{4F\,(f+\sqrt{bc})-3\Delta\right\}.$$

Thus the equation for λ becomes, on reduction,

$$\lambda^4-4\,(ch+g\sqrt{bc})\,\lambda^3+2c\,\left\{4F\,(f+\sqrt{bc})-3\Delta\right\}\lambda^2$$
$$+4\Delta c\,(ch+g\sqrt{bc})\,\lambda+\Delta^2c^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(10).$$

Putting
$$\lambda = \mu\sqrt{\Delta c},$$

we get
$$\mu^4 - \frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\,\mu^3 + 2\,\frac{4F\,(f+\sqrt{bc})-3\Delta}{\Delta}\,\mu^2$$
$$+\frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\,\mu + 1 = 0;$$

that is,
$$\left(\mu-\frac{1}{\mu}\right)^2 - \frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\left(\mu-\frac{1}{\mu}\right)$$
$$-\frac{4\,\{\Delta-2F(f+\sqrt{bc})\}}{\Delta} = 0.$$

Hence
$$\mu-\frac{1}{\mu} = \frac{2\,(f\sqrt{a}+g\sqrt{b}+h\sqrt{c}+\sqrt{abc})}{\sqrt{\Delta}} \quad\text{............}\quad (11),$$
where the ambiguities are to be taken so that
$$\sqrt{a}\,.\,\sqrt{b}\,.\,\sqrt{c} = \sqrt{abc}.$$

Solving the equation for μ, we obtain
$$\mu = \frac{f\sqrt{a}+g\sqrt{b}+h\sqrt{c}+\sqrt{abc} \pm \sqrt{2\,(\sqrt{bc}+f)(\sqrt{ca}+g)(\sqrt{ab}+h)}}{\sqrt{\Delta}}$$
$$\text{................}(12).$$

It is found, without much difficulty, that
$$\lambda^2+2H\lambda-\Delta c = 2\lambda\,(\sqrt{bc}+f)(\sqrt{ca}+g),$$
$$(\lambda^2-\Delta c)\,m_1m_2+2\lambda G = 2\lambda\,(\sqrt{bc}+f)(\sqrt{ab}+h);$$
and finally, we obtain the equations of the sixteen remaining double tangents in the form
$$\pm\sqrt{\sqrt{bc}+f}\,.\,x \pm \sqrt{\sqrt{ca}+g}\,.\,y \pm \sqrt{\sqrt{ab}+h}\,.\,z = 0,$$
where the ambiguities of the inner radicals are to be taken so that
$$\sqrt{bc}\,.\,\sqrt{ca}\,.\,\sqrt{ab} = -abc.$$

February 12th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

The President, having informed the members present of Dr. Casey's recent death, called upon Mr. Tucker to read a biographical note which had been drawn up by an intimate friend of the deceased. Dr. Larmor added a few personal reminiscences.

Mr. Tucker communicated two notes on "Isoscelians," and Mr. Heppel read a paper on "Quartic Equations interpreted by the Parabola." The Chairman read a note from Mr. W. E. Heal, of Indiana, on "the Equation of the Bitangential of the Quintic" (communicated by Prof. Cayley). Mr. Tucker read an abstract of a paper by Mr. J. Buchanan on "the Oscillations of a Spheroid in a Viscous Liquid."

The following presents were received:—

Cabinet likeness of Dr. Forsyth, F.R.S., for the Society's Album.

"Proceedings of the Royal Society," Nos. 295 and 296.

"Educational Times," for February.

"Journal of the Institute of Actuaries," Vol. XXIX., Part I.

"Transactions of the Royal Irish Academy," Vol. XXIX., Part XIV.

"Proceedings of the Royal Irish Academy," 3rd Ser., Vol. I., No. 4; Jan., 1891.

"Annals of Mathematics," Vol. V., No. 4; University of Virginia, June, 1890.

"Annual Report of the Smithsonian Institution to July, 1888," 8vo; Washington, 1890.

"Report of the Superintendent of the U.S. Naval Observatory for the year ending June 30, 1890," 8vo; Washington, 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 12; Band XV., Stück 1.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 10, 11, e 12, e Indice del Volume.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 122, and Index and Tavola Sinottica to ditto.

"Journal für die reine und angewandte Mathematik," Band CVII., Heft 3 u. 4.

"Nyt Tiddskrift for Mathematik," A., Vol. I., No. 8; B., Vol. I., No. 4; Kjobenhavn, 1890.

"Rivista di Matematica," Fasc. I., Jan., 1891; Torino, 1891.

"American Journal of Mathematics," Vol. XII., No. 2.

"The Mathematical Magazine," Vol. II., No. 3; Washington, D.C., 1891.

"An Introduction to the Logic of Algebra, with Illustrative Exercises," by Ellery W. Davis, Ph.D. (New York, Wiley & Sons); presented by Mr. Tucker.

"Elliptic Functions, an Elementary Text-book for Students of Mathematics," by A. L. Baker, Ph.D. (New York, Wiley & Sons); presented by Mr. Tucker.

Two Notes on Isoscelians. *By* R. TUCKER.

[*Read Feb.* 12*th*, 1891.]

(i.) With the in-centre, O, of ABC as centre, describe a circle to cut BC in D, D'; CA in E, E'; AB in F, F'. Then the chords DD', EE', FF' are equidistant from O, and are therefore equal to one another; also, AFE', BDF', CED' are isosceles triangles.

The lines FE', DF', ED' are *direct* Isoscelians.*

The lines DE', EF', FD', which form the triangle $\alpha\beta\gamma$, are evidently parallel to the sides of the triangle $\delta\epsilon\zeta$ ($\delta\epsilon\zeta$ being the in-circle).

Hence the set $A'B'C'$ (got by producing DF', FE', ED') and the set $\alpha\beta\gamma$ are inversely similar.

The lines $A\alpha$, $B\beta$, $C\gamma$ meet in

$$a\alpha\,(s-a) = b\beta\,(s-b) = c\gamma\,(s-c),$$

i.e., in the "Gergonne" point (G) of the triangle ABC. But $A\delta$, $B\epsilon$, $C\zeta$ meet in G; hence the set $A'B'C'$, $\alpha\beta\gamma$ are in perspective with ABC, and G is their centre of perspective.

Now
$$\angle\, \alpha Dc = 90^\circ - C/2 = \angle\, \gamma,$$

therefore $\beta\gamma$ is an antiparallel to α in the triangle $\alpha DD'$, and $A\delta$ is a median of this triangle; hence it is a Symmedian line of the triangle $\alpha\beta\gamma$, *i.e.*, G is the Symmedian point of the sets $A'B'C'$, $\alpha\beta\gamma$; and therefore *all the circles $DD'EE'FF'$ are Tucker circles of the two sets.*

(ii.) If F, F' are the *permanent centres of similitude* of the in-movable triangles abc, $a'b'c'$ of the triangle ABC (the nomenclature and the figure are adopted from *Proc.*, Vol. XVI., p. 185), and if we put

$$AcF = BaF = CbF = \phi \text{ (say)},$$
$$Bc'F' = Ca'F' = Ab'F' = \phi';$$

then
$$\left.\begin{aligned} FA &= 2\rho \sin\phi \\ F'A &= 2\rho' \sin\phi' \end{aligned}\right\} \text{ (say)},$$

and
$$bc = 2\rho \sin A,$$
$$b'c' = 2\rho' \sin A.$$

---

* The writer proposes to call the Isoscelians previously discussed by him *Oblique* Isoscelians.

Now, if (α, β, γ), $(\alpha', \beta', \gamma')$ are the trilinear coordinates of F, F', we have

$$\gamma = AF \sin FAc, \quad \beta' = AF' \sin F'Ab,$$

therefore
$$\frac{AF'}{AF} = \frac{\beta'}{\gamma} = \frac{\beta\beta'}{\beta\gamma} \propto \alpha$$

(because F, F' are isogonal conjugates).

Hence
$$\alpha \propto \frac{AF'}{AF} \propto \frac{\rho'}{\rho} \propto \frac{b'c'}{bc};$$

that is, if the sides of abc, $a'b'c'$ are respectively $p, q, r,\ p', q', r'$,

we have
$$\alpha : \beta : \gamma = \frac{p'}{p} : \frac{q'}{q} : \frac{r'}{r},$$

and
$$\alpha' : \beta' : \gamma' = \frac{p}{p'} : \frac{q}{q'} : \frac{r}{r'}.$$

The following is the extract, from Mr. Heal's letter, which was communicated to the Society by Prof. Cayley:—

"About a year ago I wrote you giving the equation of the Bitangential of the Quintic in the form

$$\left[297HJ(n, H, \phi) - 456HJ'(n, H, \phi) - 20J(n, H, \theta)\right]^2$$
$$= 3025\,(4\theta - 9H\phi)^3.$$

I showed how to reduce this equation by the well-known formula for the square of the Jacobian, and thus found the equation to be divisible by H, and all the terms except

$$\left[\left(\frac{\theta}{H}\right) - 2\theta\phi\right] H$$

were shown to be divisible by H^2.

"I have recently found the following equation, which shows that this expression is also divisible by H^2. The equation in question is

$$8\left[\left(\frac{\theta}{H}\right) - 2\theta\phi\right] = \left[9\left(\frac{\phi'}{H}\right) - 19\psi\right] H \text{..............(A)},$$

where ψ is

$$\begin{vmatrix} a, & h, & g, & H_1, & \delta_x \\ h, & b, & f, & H_2, & \delta_y \\ g, & f, & c, & H_3, & \delta_z \\ H_1, & H_2, & H_3, & & \\ \delta_x, & \delta_y, & \delta_z, & & \end{vmatrix} H.$$

I have not been able to prove this equation, but I have no doubt of its correctness.

"It is easy to show that

$$\binom{\theta}{H} - 2\theta\phi = \binom{\theta'}{H} + 2H\psi,$$

and I have tried to make the proof of equation (A) depend upon the formula

$$\dot{\square} = 8\,(8\theta - 9H\phi)$$

(see your paper on the bitangential in *Phil. Trans.*).

"Accepting equation (A) as correct, the equation of the bitangential, cleared of extraneous factors, is

$$\begin{aligned} &32400\binom{\theta}{\theta} + 376200\psi\theta - 1674675\theta\phi^2 - 1163160\theta\binom{\phi'}{H} \\ &-1259280\phi\binom{\theta}{H} + 1389960\theta\binom{\phi}{H} \\ &+\Big[2205225\phi^3 - 6014250\phi\binom{\phi}{H} + 20516544\phi\binom{\phi'}{H} \\ &\qquad\qquad -962280\binom{\theta}{\phi} + 1480680\binom{\theta}{\phi'}\Big]H \\ &+\Big[7144929\binom{\phi}{\phi} - 21939984\binom{\phi}{\phi'} + 16701816\binom{\phi'}{\phi'}\Big]H^2 = 0. \end{aligned}$$

"A remarkable property of the function ψ is that, regarding it as a quadric function in $\delta_x, \delta_y, \delta_z$, the discriminant vanishes, or it breaks up into two linear factors. I am now working to prove equation (A), which will complete the proof of the above equation of the bitangential."

The Oscillations of a Spheroid in a Viscous Liquid.

By J. BUCHANAN.

[*Read February 12th,* 1891.]

The solution of the problem relating to the motion of spherical and cylindrical pendulums, which perform small oscillations in a viscous liquid, was obtained by Sir George Stokes in a paper published in the *Camb. Phil. Transactions* for 1850. In a joint memoir by Helmholtz and Piotrowski, the torsional oscillations of spheres and cylinders have been discussed. But the only problems relating to the motion of spheroids which have been solved are those of steady motion.

The equations in the case of small oscillations are analogous to those of the conduction of heat. These have been discussed by M. Mathieu (*Cours de Physique Math.*), and Prof. C. Niven (*Phil. Trans.*, 1880). Choosing with them as our coordinates, ϕ the azimuth of the meridian section, and α and β the parameters of the confocal ellipsoids and hyperboloids of revolution which pass through the point, the velocity round the axis can be expressed in a series of terms of the form

$$\epsilon^{-\lambda^2\nu t} \,.\, \vartheta_k(\cos\beta) \,.\, \Omega_\kappa(\lambda c\cosh\alpha),$$

where k is determined by an equation whose roots are infinite in number. The function ϑ is expanded in a series of associated functions of the first kind of the first order and odd degrees; while Ω is expressed as a series of spherical functions of $\lambda c\cosh\alpha$, $c\cosh\alpha$ being the polar axis of the spheroid, and also in terms of associated functions of the second kind. The couple on the rotating spheroid is then obtained.

Approximate values are found when the ellipticity of the spheroid is small; and the corresponding equation of motion of the sphere obtained. When the spheroid becomes a sphere, the results agree with those already known. The solution for a spheroid rotating steadily is next written down; and from it is deduced that for a disc.

In the second part of the paper the motion of a spheroidal pendulum, oscillating along its axis, is discussed. We use Stokes' stream line function, and find that the value of ψ consists of two parts. The first of these consists of terms of the form

$$e^{-\lambda^2\nu t} . \sin\beta \sinh \alpha P_n^1 (\cos\beta) . Q_n^1 (\cosh\alpha),$$

where P_n^1, Q_n^1 are associated functions of the first and second kinds of the first order and odd degree. The other part can be expressed as a series of terms of the form

$$e^{-\lambda^2\nu t} . \sin\beta \sinh \alpha \vartheta (\mu) . \Omega (\eta),$$

where ϑ and Ω have the same values as before. The resistance exerted by the liquid is also found, and the motion of the spheroid, when the ellipticity is small, is considered.

These results have been obtained in the first instance for a prolate spheroid; but they are applicable, with slight alterations, to the case of a spheroid whose axis of revolution is the least axis.

Torsional Oscillations of a Spheroid in a Viscous Fluid.

1. Let us suppose the spheroid to be suspended in a viscous fluid by means of a torsion fibre, and to oscillate about its axis. If the fluid be initially at rest, or move in annuli round the axis of symmetry, the motion will always be in annuli round the axis.

Starting with cylindrical coordinates, we have

$$u = 0, \quad w = 0,$$

and p the pressure, and v the velocity round the axis of symmetry, are independent of ϕ.

Neglecting squares and products of velocities, two of the equations of motion give us

$$\frac{dQ}{d\varpi} = 0, \quad \frac{dQ}{dz} = 0,$$

where $$Q = -V - p/\rho = gz - p/\rho,$$

if gravity alone act.

These equations show that Q is independent of ϖ and z, and by symmetry it must be independent of ϕ; hence we must take Q equal to a constant or zero.

The third equation of motion reduces to

$$\left(\nabla^2 - \frac{1}{\varpi^2}\right) v = \frac{1}{\nu} \frac{dv}{dt} \quad \text{.........................(1)},$$

where
$$\nabla^2 = \frac{d^2}{dz^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\,\frac{d}{d\varpi},$$

and ν is the "kinematical coefficient of viscosity."

Since the fluid moves in annuli round the axis, we might put

$$v = \varpi\,.\,\omega e^{-\lambda^2\nu t},$$

where $\omega e^{-\lambda^2\nu t}$ is the angular velocity of the fluid at distance ϖ from the axis; ϖ will thus appear as a factor of v in the solution of the above equation.

2. Let us take as our standard case that of a prolate spheroid. Writing

$$x = c\sinh\alpha\sin\beta\cos\phi,\quad y = c\sinh\alpha\sin\beta\sin\phi,$$

$$z = c\cosh\alpha\cos\beta,$$

or
$$\left.\begin{aligned}\varpi = \sqrt{x^2+y^2} &= c\sinh\alpha\sin\beta,\\ z &= c\cosh\alpha\cos\beta,\end{aligned}\right\}$$

α, β, ϕ form a new orthogonal system of coordinates—the orthogonal surfaces being confocal spheroids and hyperboloids of two sheets and planes passing through the axis of symmetry.

Also, if u, v, w be the velocities along the normals to these surfaces,

$$w = v \text{ of cylindrical coordinates.}$$

Transforming to these coordinates, equation (1) becomes*

$$\frac{d^2w}{d\alpha^2} + \frac{d^2w}{d\beta^2} + \coth\alpha\frac{dw}{d\alpha} + \cot\beta\frac{dw}{d\beta} - \left(\frac{1}{\sinh^2\alpha} + \frac{1}{\sin^2\beta}\right)w$$
$$= \frac{c^2}{\nu}(\cosh^2\alpha - \cos^2\beta)\frac{dw}{dt} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The surface condition, when the fluid sticks, is

$$w = \varpi_0\omega_0 e^{-\lambda^2\nu t} = c\sinh\alpha_0\sin\beta\,.\,\omega_0 e^{-\lambda^2\nu t} \quad\ldots\ldots\ldots\ldots\ldots(3),$$

where ϖ_0, α_0 are the surface values of ϖ, α; and $\omega_0 e^{-\lambda^2\nu t}$ is the angular velocity of the spheroid.

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* *Cf.* Niven "On the Conduction of Heat in Ellipsoids of Revolution," *Phil. Trans.*, Pt. I., 1880, p. 122.

If slipping be supposed to exist, the surface condition is

$$B\left[w - c\sinh\alpha_0\sin\beta\,.\,\omega_0\epsilon^{-\lambda^2\nu t}\right] = \text{tangential stress in direction opposite to the motion}$$

$$= p_{\alpha\phi} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}',$$

where B is the coefficient of sliding friction.

If the fluid extend to infinity, we have the additional condition that the velocity vanishes there; if it be internal, the velocity must not become infinite at the origin; if it be limited internally and externally, another condition similar to (3) or (3)′ must be satisfied at the other bounding surface.

If we take a small element bounded by consecutive pairs of surfaces, its edges are

$$ds_1 = h^{-1}d\alpha, \quad ds_2 = h^{-1}d\beta, \quad ds_3 = \varpi\, d\phi,$$

where

$$h^2 = \frac{d(\alpha\beta)}{d(z\varpi)} = \frac{1}{c^2(\cosh^2\alpha - \cos^2\beta)}.$$

Then *uvw* are the velocities along the edges of this element, and if

$p_{\alpha\alpha}$ represent the stress on the face α in the direction in which α increases,

$p_{\alpha\beta}$ represent the stress on the face α in the direction in which β increases,

or represent the stress on the face β in the direction in which α increases, &c.,

the values of these stresses* are given by the equations

$$\left.\begin{aligned} p_{\alpha\alpha} &= -p + 2\mu h\left[\frac{du}{d\alpha} - \frac{v}{h}\,.\,\frac{dh}{d\beta}\right] \\ p_{\beta\beta} &= -p + 2\mu h\left[\frac{dv}{d\beta} - \frac{u}{h}\,.\,\frac{dh}{d\alpha}\right] \\ p_{\phi\phi} &= -p + 2\mu h\left[\frac{u}{\varpi}\,.\,\frac{d\varpi}{d\alpha} + \frac{v}{\varpi}\,.\,\frac{d\varpi}{d\beta}\right] \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\text{(4)},$$

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* *Cf.* Ibbetson, *Elasticity*, p. 250, eqns. 127.

and

$$\left.\begin{aligned} p_{\alpha\beta} &= \mu\left[\frac{d}{d\alpha}(hv)+\frac{d}{d\beta}(hu)\right] \\ p_{\beta\phi} &= \mu h\varpi\frac{d}{d\beta}\left(\frac{w}{\varpi}\right) \\ p_{\alpha\phi} &= \mu h\varpi\frac{d}{d\alpha}\left(\frac{w}{\varpi}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $\mu = \nu\rho =$ the coefficient of viscosity.

3. Assuming the time to enter in the form of the factor $\epsilon^{-\lambda^2\nu t}$, we can satisfy equation (2) by writing

$$w = \epsilon^{-\lambda^2\nu t}\,.\,f(\alpha)\,.\,F(\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where f, F are solutions of the equations

$$\left.\begin{aligned} \frac{d^2F}{d\beta^2}+\cot\beta\,\frac{dF}{d\beta}-\frac{F}{\sin^2\beta} &= (\lambda^2c^2\cos^2\beta-k)\,F, \\ \frac{d^2f}{d\alpha^2}+\coth\alpha\,\frac{df}{d\alpha}-\frac{f}{\sinh^2\alpha} &= (k-\lambda^2c^2\cosh^2\alpha)\,f, \end{aligned}\right\}$$

k being a constant, which has an infinite number of values corresponding to one value of λ.

If we write μ for $\cos\beta$, and η for $\lambda c\cosh\alpha$, these equations become —

$$\left.\begin{aligned} (1-\mu^2)\frac{d^2F}{d\mu^2}-2\mu\frac{dF}{d\mu}-\frac{F}{1-\mu^2} &= (\lambda^2c^2\mu^2-k)\,F \\ (\eta^2-\lambda^2c^2)\frac{d^2f}{d\eta^2}+2\eta\frac{df}{d\eta}-\frac{\lambda^2c^2}{\eta^2-\lambda^2c^2}\,f &= (k-\eta^2)\,f \end{aligned}\right\} \ldots\ (7, 8).$$

The first of these equations can be satisfied by a series of associated functions of the first order. Denoting by P_1^n, Heine's associated function* of the first order and n^{th} degree, we

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* Heine's associated function of the m^{th} order and n^{th} degree is denoted by P_m^n, and is equal to

$$\frac{2^n\,.\,n-m\,!\,n\,!}{2n\,!}(\mu^2-1)^{\frac{1}{2}m}\frac{d^mP_n}{d\mu^m},$$

the constant being so chosen that the coefficient of the highest power of μ is unity. If we use the notation P_n^m, where

$$P_n^m = (1-\mu^2)^{\frac{1}{2}m}\frac{d^mP_n}{d\mu^m},$$

have*
$$\mu P_1^n = P_1^{n+1} + \frac{n^2-1}{4n^2-1} P_1^{n-1};$$

therefore
$$\mu^2 P_1^n = P_1^{n+2} + \frac{2n^2+2n-3}{(2n-1)(2n+3)} P_1^n + \frac{n^2-1}{4n^2-1} \cdot \frac{(n-1)^2-1}{4(n-1)^2-1} P_1^{n-2} \quad \text{..............} (9).$$

Also, since the value of w is unaltered when β changes to $\pi-\beta$, and therefore μ into $-\mu$, it must contain only even powers of μ, and therefore only associated functions of the first order and *odd* degree.

In Heine's notation the solution† is

$$\vartheta(\mu) = c\left[a_0 P_1^1 - a_1 P_1^3 + \ldots + (-1)^r a_r P_1^{2r+1} + \ldots\right],$$

or, using the notation P_n^m for the associated function of the m^{th} order and n^{th} degree, P_n^m being equal to

$$(1-\mu^2)^{\frac{1}{2}m} \frac{d^m P_n}{d\mu^m},$$

this becomes

$$\vartheta(\mu) = c'\left[a_0 P_1^1 - \frac{2}{3.5} a_1 P_3^1 + \ldots + (-1)^r \frac{1.2\ldots\overline{n-1}}{1.3\ldots\overline{2n-1}} a_r P_n^1 + \ldots\right] \quad \text{...........} (10),$$

---

this is connected with Heine's function by the relation

$$P_m^n = i^m \cdot \frac{2^n \cdot \overline{n-m}!\, n!}{2n!} P_n^m.$$

Similarly, if
$$Q_n^m = (\zeta^2-1)^{\frac{1}{2}m} \frac{d^m Q_n}{d\zeta^m}$$

be defined as the associated function of the second kind of the m^{th} order and n^{th} degree,

$$Q_m^n = (-i)^m \cdot \frac{\overline{2n+2}!}{2^{n+1} \cdot \overline{n+m}!\, \overline{n+1}!} Q_n^m.$$

With this notation the first and fifth equations in Heine's *Kugel-Functionen*, (Vol. I., p. 258) become

$$\left.\begin{aligned}(2n+1)\mu P_n^m &= (n-m+1) P_{n+1}^m + (n+m) P_{n-1}^m, \\ (2n+1)\mu Q_n^m &= (n-m+1) Q_{n+1}^m + (n+m) Q_{n-1}^m.\end{aligned}\right\}$$

* Heine, *Kugel-Functionen*, Vol. I., p. 258, eqn. 1.

† *Phil. Trans.*, 1880, Pt. I., pp. 124, 125.

where $a_0, a_1, \dots a_n$ are connected by the equations

$$\left.\begin{aligned} p_1 a_1 &= \frac{a_0}{\lambda^2 c^2}(\kappa_0 - k) \\ p_2 a_2 &= \frac{a_1}{\lambda^2 c^2}(\kappa_1 - k) - a_0 \\ \dots \quad & \dots \quad \dots \quad \dots \\ p_{r+1} a_{r+1} &= \frac{a_r}{\lambda^2 c^2}(\kappa_r - k) - a_{r-1} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots (11),$$

in which

$$p_r = \frac{n^2-1}{4n^2-1} \cdot \frac{(n-1)^2-1}{4(n-1)^2-1},$$

$$\kappa_r = n(n+1) + \frac{2n^2+2n-3}{(2n-1)(2n+3)} \lambda^2 c^2, \qquad n = 2r+1.$$

The solution of the equation

$$(\eta^2 - \lambda^2 c^2)\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} - \frac{\lambda^2 c^2}{\eta^2 - \lambda^2 c^2} f = (k - \eta^2) f \dots\dots (8)$$

can be expressed* in terms of a series of spherical functions, S_0, $S_1 \dots$, $S_n \dots$, where S_n is the solution of the equation

$$\frac{d^2 S_n}{d\eta^2} + \frac{2}{\eta}\frac{dS_n}{d\eta} + \left(1 - \frac{n \, . \, \overline{n+1}}{\eta^2}\right) S_n = 0 \quad \dots\dots\dots\dots (12).$$

The solution is

$$f(\alpha) = \Omega(\eta)$$

$$= \frac{(\eta^2 - \lambda^2 c^2)^{\frac{1}{2}}}{\eta}\left[2a_0 S_1 + \frac{8}{5} a_1 S_3 + \dots + \frac{1 \, . \, 2 \dots \overline{n+1}}{1 \, . \, 3 \dots \overline{2n-1}} a_r S_n + \dots\right] \dots (13),$$

where

$$n = 2r+1,$$

and the coefficients are determined by the same set of equations as before.

If the fluid extend to infinity, we take the solution

$$S_n = \eta^n \left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\epsilon^{-i\eta}}{\eta}.$$

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* *Phil. Trans.*, 1880, Pt. I., pp. 128–130.

If the fluid be internal, we write

$$S_n = \eta^n \left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\sin\eta}{\eta}.$$

If it be limited externally and internally, we use the general form of S_n, obtained by combining these two solutions.

It is easy to show* that

$$\left.\begin{aligned} \eta^{n+1}\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\epsilon^{-i\eta}}{\eta} &= (-i)^n.\epsilon^{-i\eta}\left[1+\frac{n+1.n}{2.i\eta}+\frac{(n+2)\ldots(n-1)}{2.4(+i\eta)^2}+\ldots\right] \\ &= (-i)^n\,\epsilon^{-i\eta} f_n(i\eta) \\ \eta^{n+1}\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\epsilon^{i\eta}}{\eta} &= (i)^n.\,\epsilon^{i\eta}\left[1+\frac{n+1.n}{2(-i\eta)}+\frac{(n+2)\ldots(n-1)}{2.4(-i\eta)^2}+\ldots\right] \\ &= (i)^n\,\epsilon^{i\eta} f_n(-i\eta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

The solution of equation (8) can be got in another form by writing ζ for $\cosh\alpha$. In this case the equation becomes

$$(\zeta^2-1)\frac{d^2f}{d\zeta^2}+2\zeta\frac{df}{d\zeta}-\frac{1}{\zeta^2-1}f = (k-\lambda^2c^2\zeta^2)f \quad \ldots\ldots\ldots(15),$$

which is of the same form as (7).

If we want a solution which vanishes at infinity, this will involve harmonics of the type Q_n^1. Now, since the relation between Q_n^1, Q_{n+1}^1, Q_{n-1}^1 is of the same form as that between μP_n^1, P_{n+1}^1, P_{n-1}^1, the solution of this equation is

$$\phi(\zeta) = c'\left[a_0Q_1^1-\frac{2}{3.5}a_1Q_3^1+\ldots+(-1)^r\frac{1.2\ldots n-1}{1.3\ldots 2n-1}a_rQ_n^1+\ldots\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(16),$$

where $$n = 2r+1.$$

4. Equations (11) have been discussed by Prof. Niven in the memoir already referred to. There it is shown that, treating a_0 as positive throughout, the series of coefficients $a_0, a_1, \ldots a_r$ gains r changes of sign as k passes from $-\infty$ to $+\infty$. Hence all the r roots of $a_r = 0$ are real. Further, all the roots of $a_r = 0$ lie below κ_r; and

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* *Cf.* Lord Rayleigh, *Sound*, Vol. II., Chap. XVII.

one, and only one, lies between κ_r and κ_{r-1}. Also, each of the equations $a_{r+1} = 0$, $a_{r+2} = 0$, ... has one root between each pair of roots of the equation $a_r = 0$. Thus each of the coefficients $a_0, a_1, \ldots$ vanishes once, and only once, for certain values of k lying between $-\infty, \kappa_0, \kappa_1 \ldots, +\infty$; and the values for which a_r vanishes, when r is very great, converge to certain definite values. In the neighbourhood of these, the coefficients converge with great rapidity.

Thus the values of k are all real, and are the roots of the equation

$$a_\infty = 0.$$

For these values the series of coefficients is rapidly convergent. If $k_0, k_1, \ldots$ be the values of k obtained from this equation,

$$w = \Sigma\, \epsilon^{-\lambda^2\nu t} A_k \vartheta_k(\mu)\, \Omega_k(\eta)$$

$$= \Sigma_{r=0}^{r=\infty} \epsilon^{-\lambda^2\nu t} A_r \left[a_{r0} P_1^1 - \frac{2}{3.5} a_{r1} P_3^1 + \ldots\right] \Omega_r(\eta),$$

where $a_{r0}, a_{r1}, \ldots a_{rr}$ are the coefficients corresponding to the root k_r. Rearranging, this takes the form

$$w = \epsilon^{-\lambda^2\nu t} \Sigma_{r=0}^{r=\infty} (-1)^r \cdot \frac{1.2 \ldots \overline{n-1}}{1.3 \ldots \overline{2n-1}} \left[A_0 a_{0r} \Omega_0(\eta) + A_1 a_{1r} \Omega_1(\eta) + \ldots\right] P_n^1$$

$$= \epsilon^{-\lambda^2\nu t} \Sigma_{r=0}^{r=\infty} (-1)^r \cdot \frac{2^n . n!\, \overline{n-1}!}{2n!} \cdot \frac{(\eta^2 - \lambda^2 c^2)}{\eta} \left[2 b_{0r} S_1 + \tfrac{8}{5} b_{1r} S_3 + \ldots\right] P_n^1$$

$$= \epsilon^{-\lambda^2\nu t} \Sigma_{r=0}^{r=\infty} \Sigma_{s=0}^{s=\infty} (-1)^r \cdot \frac{2^{m+n} . n!\, \overline{n-1}!\, m!\, (\overline{m+1}!)^m}{2n!\, 2m!} b_{sr} \tanh \alpha S_m \,.\, P_n^1 \qquad \ldots\ldots\ldots\ldots\ldots(17),$$

where $$n = 2r+1, \quad m = 2s+1,$$

and $$b_{sr} = A_0 a_{0r} a_{0s} + A_1 a_{1r} a_{1s} + \ldots + A_r a_{rr} a_{rs} + \ldots .$$

In terms of associated functions of the first and second kinds, the solution is

$$w = \epsilon^{-\lambda^2\nu t} \Sigma_{r=0}^{r=\infty} (-1)^r . \frac{2^n . n!\, \overline{n-1}!}{2n!} \left[b_{0r} Q_1^1 - \frac{2}{3.5} b_{1r} Q_3^1 + \ldots + \ldots\right] P_n^1$$

$$= \epsilon^{-\lambda^2\nu t} \Sigma_{r=0}^{r=\infty} \Sigma_{s=0}^{s=\infty} (-1)^{r+s} \frac{2^{m+n} . n!\, \overline{n-1}!\, m!\, \overline{m-1}!}{2n!\, 2m!} b_{sr} Q_m^1 \,.\, P_n^1 \ldots(18).$$

At the surface of the spheroid, if the fluid stick,

$$w = c \sinh \alpha_0 \sin \beta \omega_0 \epsilon^{-\lambda^2\nu t}.$$

Hence the equations which determine the arbitrary constants are

$$\left.\begin{array}{l} 2b_{00}S_1(\eta_0)+\frac{8}{5}b_{10}S_3(\eta_0)+\ldots = c\omega_0\cosh\alpha_0 \\ 2b_{01}S_1(\eta_0)+\frac{8}{5}b_{11}S_3(\eta_0)+\ldots = 0 \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \end{array}\right\}\ldots\ldots\ldots\ldots(19).$$

If slipping take place, the surface condition is

$$B\left[w-c\sinh\alpha_0\sin\beta\,.\,\omega_0\,.\,\epsilon^{-\lambda^2\nu t}\right]=p_{\alpha\phi}=\nu\rho\varpi h\frac{d}{d\alpha}\left(\frac{w}{\varpi}\right)_{\alpha=\alpha_0}$$

$$=\frac{\nu\rho\sinh\alpha_0}{c\sqrt{\cosh^2\alpha_0-\cos^2\beta}}\frac{d}{d\alpha_0}\left(\frac{w}{\sinh\alpha_0}\right).$$

If a, b be the semi-axes of the elliptic section, e the eccentricity,

$$\mathrm{a}=c\cosh\alpha_0,\quad \mathrm{b}=c\sinh\alpha_0,\quad e=\operatorname{sech}\alpha_0.$$

Hence the above may be written in the form

$$\Sigma_0^\infty(-1)^r\,.\,\frac{1\,.\,2\ldots\overline{n-1}}{1\,.\,3\ldots\overline{2n-1}}\tanh\alpha_0\left[2b_{0r}S_1+\frac{8}{5}b_{1r}S_3+\ldots\right]P_n^1(\mu)$$
$$-c\omega_0\sinh\alpha_0\sin\beta$$

$$=\frac{\lambda^2\nu\rho\mathrm{b}^2}{B\mathrm{a}\sqrt{1-e^2\mu^2}}\Sigma_0^\infty(-1)^r\frac{1\,.\,2\ldots(n-1)}{1\,.\,3\ldots(2n-1)}\frac{d}{d\eta_0}$$
$$\times\left[\frac{1}{\eta_0}\left\{2b_{0r}S_1+\frac{8}{5}b_{1r}S_3+\ldots\right\}\right]P_n^1(\mu).\quad(n=2r+1.)$$

Now $(1-e^2\mu^2)^{-\frac{1}{2}}$ can be expanded in a convergent series of even powers of μ, and we have already seen how to express $\mu^2P_n^1$, $\mu^4P_n\ldots$ in terms of associated functions of the same type as P_n^1. Hence $(1-e^2\mu^2)^{-\frac{1}{2}}P_n^1(\mu)$ can be expressed in terms of associated functions of the form $P_{2r+1}^1(\mu)$; let the expression be

$$c_{r0}P_1^1+c_{r1}P_3^1+\ldots+c_{rr}P_{2r+1}^1+\ldots.$$

Then the constants are determined by the equations

$$\left.\begin{array}{l} 2b_{00}S_1(\eta_0)+\frac{8}{5}b_{10}S_3(\eta_0)+\ldots \\ \qquad=c\omega_0\cosh\alpha_0+\frac{\lambda^2\nu\rho\mathrm{b}}{B\mathrm{a}}\Sigma_1^\infty(-1)^r c_{r0}\frac{d}{d\eta_0}\left[\frac{1}{\eta_0}\left\{2b_{0r}S_1(\eta_0)+\ldots\right\}\right] \\ 2b_{10}S_1(\eta_0)+\frac{8}{5}b_{11}S_3(\eta_0)+\ldots \\ \qquad=\qquad\frac{\lambda^2\nu\rho\mathrm{b}}{B\mathrm{a}}\Sigma_1^\infty(-1)^r c_{r1}\frac{d}{d\eta_0}\left[\frac{1}{\eta_0}\left\{2b_{0r}S_1+\ldots\right\}\right] \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \end{array}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

The corresponding equations involving associated functions of the type $Q_n^1(\zeta_0)$ may easily be written down from equation (18).

5. We get the couple resisting the motion of the spheroid by taking moments about the axis of rotation. This gives

$$G = -\iint p_{a\phi}\, dS \,.\, \varpi,$$

where the integration extends all over the spheroid, *i.e.*, from $\phi = 0$ to 2π, and $\beta = 0$ to π. Thus

$$\begin{aligned} G &= -\int_0^{2\pi}\int_0^{\pi} \nu\rho\varpi h \frac{d}{da}\left(\frac{w}{\varpi}\right)\varpi d\phi \,.\, h^{-1} d\beta \,.\, \varpi \\ &= -2\pi\nu\rho\epsilon^{-\lambda^2\nu t}\int_{-1}^{+1} c^3 \sinh^3 a_0 \,.\, \frac{d}{da_0}\left(\frac{w}{\sinh a_0}\right) . P_1^1(\mu)\, d\mu \\ &= -2\pi\nu\rho\lambda^2\epsilon^{-\lambda^2\nu t} c^4 \sinh^4 a_0 \frac{d}{d\eta_0} \\ &\qquad\qquad \times\left[\frac{1}{\eta_0}\left\{2b_{00}S_1 + \frac{8}{5} b_{01}S_3 + \dots\right\}\right]_{\eta=\eta_0}\int_{-1}^{+1}(P_1^1)^2\, d\mu \\ &= -\tfrac{8}{3}\pi\rho\lambda^2\nu\epsilon^{-\lambda^2\nu t} \,.\, \mathrm{b}^4 \Sigma_{r=0}^{r=\infty} \frac{1\,.\,2 \dots \overline{n+1}}{1\,.\,3 \dots \overline{2n-1}}\, \mathrm{b}_{0r} \,.\, \frac{1}{\eta} \\ &\qquad\qquad \times\left[S_{n+1} + (n-1)\,.\,\frac{S_n}{\eta}\right]^{*}_{\eta=\eta_0} \quad \dots\dots\dots\dots (21); \end{aligned}$$

where, as before, $n = 2r+1$.

6. We can find approximate values for w and G when the ellipticity of the spheroid is so small that terms involving the cube of λ^2c^2 may be neglected. Since, when $e = 0$, the spheroid becomes a sphere and $k = n(n+1)$, we assume

$$k = n(n+1) + \mu_1\lambda^2c^2 + \mu_2\lambda^4c^4 + \dots .$$

The equations giving the ratios of the coefficients a_0, a_1, ... are

$$\lambda^2c^2 \,.\, \frac{p_1a_1}{a_0} = \kappa_0 - k,$$

$$\lambda^2c^2 \,.\, \frac{p_2a_2}{a_1} = \kappa_1 - k - \lambda^2c^2 \,.\, \frac{a_0}{a_1},$$

$$\lambda^2c^2 \,.\, \frac{p_3a_3}{a_2} = \kappa_2 - k - \lambda^2c^2 \,.\, \frac{a_1}{a_2}, \text{ and so on.}$$

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* *Cf. Phil. Trans.*, 1880, Part I., p. 128.

When $$r = 0, \quad n = 1,$$

and the first approximate value of k is κ_0, or

$$1 . 2 + \tfrac{1}{5}\lambda^2c^2.$$

A second approximation is got from the second equation by writing

$$\kappa_1 - k - \lambda^2c^2 . \frac{a_0}{a_1} = 0,$$

therefore $$\kappa_1 - k - \lambda^2c^2 . \frac{p_1\lambda^2c^2}{\kappa_0 - k} = 0,$$

or $$k = \kappa_0 - \lambda^4c^4 . \frac{p_1}{\kappa_1 - k},$$

where, in the term multiplied by λ^4c^4, we write for k its first approximate value.

Now $$p_1 = \frac{8}{5^2 . 7}, \quad \kappa_1 - \kappa_0 = 10 + \frac{4}{15}\lambda^2c^2.$$

Hence the second approximation to the value of k is

$$2 + \tfrac{1}{5}\lambda^2c^2 - \frac{4}{5^3 . 7}\lambda^4c^4 + \frac{8}{3 . 5^5 . 7}\lambda^6c^6,$$

and the ratios of the coefficients corresponding to this value are

$$\frac{a_{01}}{a_{00}} = \frac{\kappa_0 - k}{p_1\lambda^2c^2} = \frac{1}{10}\lambda^2c^2 - \frac{1}{3 . 5^3}\lambda^4c^4, \quad \frac{a_{02}}{a_{00}} = \frac{\lambda^2c^2}{\kappa_2 - k} . \frac{1}{10}\lambda^2c^2 = \frac{\lambda^4c^4}{280} \quad \text{......(22).}$$

In the same way, when

$$r = 1, \quad n = 3,$$

the first approximate value of k is κ_1,

or $$12 + \tfrac{7}{15}\lambda^2c^2;$$

the second approximation is got by writing

$$k = \kappa_1 - \lambda^2c^2\left(\frac{p_1\lambda^2c^2}{\kappa_0 - \kappa_1} + \frac{p_2\lambda^2c^2}{\kappa_2 - \kappa_1}\right).$$

And the ratios of the coefficients are

$$\frac{a_{10}}{a_{11}} = +\frac{p_1\lambda^2c^2}{\kappa_0 - k} = -\frac{4}{5^3 . 7}\lambda^2c^2 + \frac{8}{3 . 5^5 . 7}\lambda^4c^4, \quad \frac{a_{12}}{a_{11}} = \frac{\lambda^2c^2}{\kappa_2 - k} = \frac{\lambda^2c^2}{18} - \frac{\lambda^4c^4}{3^5 . 5 . 13} \quad \text{.................(23).}$$

It is evident, also, that A_0 differs from a constant by a term of the order λ^2c^2; A_1 is of order λ^2c^2; and so on. Hence, if we keep terms up *to* λ^4c^4, *and treat* a_{00}, a_{11}, a_{22} as unity, the equations which determine the coefficients A_0, A_1, A_2 are

$$\left.\begin{array}{l} A_0 \left(S_1+\frac{4}{5}a_{01}S_3+\frac{8}{21}a_{02}S_5\right)+\frac{4}{5}A_1a_{10}S_3=\frac{1}{2}c\omega_0\cosh a_0 \\ A_0a_{01}\left(S_1+\frac{4}{5}a_{01}S_3\right)+A_1\left[a_{10}S_1+\frac{4}{5}S_3+\frac{8}{21}a_{12}S_5\right]=0 \\ A_0a_{02}\,S_1+\qquad A_1\,.\,\frac{4}{5}a_{12}S_3+\frac{8}{21}A_2S_5 \qquad =0 \end{array}\right\}.$$

The solutions of these equations, to our order of approximation, are

$$\left.\begin{array}{l} A_0=\frac{1}{2}\frac{c\omega_0\cosh a_0}{S_1}\left[1-\frac{4}{5}a_{01}\frac{S_3}{S_1}+a_{01}a_{10}+\frac{16}{25}a_{01}^2\frac{S_3^2}{S_1^2}-\frac{8}{21}a_{02}\frac{S_5}{S_1}\right] \\ A_1= \qquad -\frac{5}{8}a_{01}\,.\,\frac{c\omega_0\cosh a_0}{S_3}\left[1-\frac{5}{4}a_{10}\frac{S_1}{S_3}-\frac{10}{21}a_{12}\frac{S_5}{S_3}\right] \\ A_2 \qquad -\frac{21}{16}(a_{02}-a_{01}a_{12})\frac{c\omega_0\cosh a_0}{S_5} \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(24),$$

where, in the expressions for S_1, S_3, &c., ... η_0 is written for η.

To the same order of small quantities

$$w=2e^{-\lambda^2\nu t}\tanh a\left[\left\{A_0\left(S_1+\frac{4}{5}a_{01}S_3+\frac{8}{21}a_{02}S_5\right)+\frac{4}{5}A_1a_{10}S_3\right\}P_1^1\right.$$

$$-\frac{2}{3\,.\,5}\left\{A_0a_{01}\left(S_1+\frac{4}{5}a_{01}S_3\right)+A_1\left(a_{10}S_1+\frac{4}{5}S_3+\frac{8}{21}a_{12}S_5\right)\right\}P_3^1$$

$$\left.+\frac{2\,.\,4}{5\,.\,7\,.\,9}\left\{A_0a_{02}S_1+A_1\frac{4}{5}a_{12}S_3+\frac{8}{21}A_2S_5\right\}P_5^1\right] \quad\ldots\ldots\ldots\ldots\ldots(25).$$

Also, from (21), the couple is

$$G=-\frac{8}{3}\pi\rho\lambda^2\nu\epsilon^{-\lambda^2\nu t}\,.\,\frac{\mathrm{b}^4}{\lambda a}$$

$$\times\left[2A_0S_2+\frac{8}{5}(A_0a_{01}+A_1a_{10})\left(S_4+\frac{2}{\lambda a}S_3\right)+\frac{16}{21}A_0a_{02}\left(S_6+\frac{4}{\lambda a}S_5\right)\right]$$

$$=-\frac{8}{3}\pi\rho\lambda\nu\omega_0e^{-\lambda^2\nu t}\,.\,\mathrm{b}^4$$

$$\times\left[\frac{S_2}{S_1}-\frac{4}{5}a_{01}\left(\frac{S_2S_3}{S_1^2}-\frac{S_4}{S_1}-\frac{2}{\lambda a}\cdot\frac{S_3}{S_1}\right)+a_{01}a_{10}\left(\frac{S_2}{S_1}-\frac{S_4}{S_3}-\frac{2}{\lambda a}\right)\right.$$

$$\left.+\frac{16}{25}a_{01}^2\frac{S_3}{S_1}\left(\frac{S_2S_3}{S_1^2}-\frac{S_4}{S_1}-\frac{2}{\lambda a}\cdot\frac{S_3}{S_1}\right)-\frac{8}{21}a_{02}\left(\frac{S_2S_5}{S_1^2}-\frac{S_6}{S_1}-\frac{4}{\lambda a}\cdot\frac{S_5}{S_1}\right)\right].$$

Now* $$S_{n+1}+\frac{2n+1}{\lambda a}S_n+S_{n-1}=0,$$

therefore $$\lambda a\,(S_2S_3-S_1S_4)-2S_1S_3=-5\left[S_1^2+S_2^2+\frac{5S_1S_2}{\lambda a}\right],$$

and $$\lambda a\,(S_2S_5-S_1S_6)-4S_1S_5$$

$$=\lambda a\,(S_1S_4-S_2S_3)+2S_1S_3-9\left[S_1S_3+S_4\left(S_2+\frac{7}{\lambda a}S_1\right)\right]$$

$$=+5\left[S_1^2+S_2^2+\frac{5S_1S_2}{\lambda a}\right]+9\left(1-\frac{49}{\lambda^2a^2}\right)\left(S_1^2+S_2^2+\frac{5S_1S_2}{\lambda a}\right)+\frac{126}{\lambda^2a^2}S_2^2.$$

Hence the part of the expression for G within the square brackets may be written

$$\frac{S_2}{S_1}+\frac{5S_1S_2+\lambda a\,(S_1^2+S_2^2)}{\lambda^2a^2S_1^2}$$

$$\times\left[4a_{01}-5a_{01}a_{10}\frac{S_1}{S_3}-\frac{16}{5}a_{01}^2\frac{S_3}{S_1}-\frac{8}{3}a_{02}\left(2-\frac{7.9}{\lambda^2a^2}\right)\right]-\frac{8.6}{\lambda^2a^2}\cdot a_{02}S_2^2 \qquad \text{..............(26).}$$

The terms involving e^2 are complicated, and would be troublesome to evaluate; however, if we refer to the values of the coefficients a_{01}, a_{02}, &c., which we have obtained [equations (22) and (23)], we see that the numerical factors in these expressions are very small. Also, it will appear afterwards that the expression $\dfrac{5S_1S_2+\lambda a\,(S_1^2+S_2^2)}{S_1^2}$ is not a large quantity. Hence only a very small error will probably be introduced if we neglect the terms of the second order.

If the fluid be external, equation (14) gives us

$$S_1(\lambda a)=-e^{-i\lambda a}\cdot\frac{1+i\lambda a}{\lambda^2a^2},\quad S_2(\lambda a)=e^{-i\lambda a}\cdot\frac{3+3i\lambda a+(i\lambda a)^2}{\lambda^3a^3}.$$

Hence, if m be the mass of the fluid displaced, we find after a few reductions $$G=2m\nu\omega_0e^{-\lambda^2\nu t}\cdot\frac{b^2}{a^2}$$

$$\times\left[\frac{3+3i\lambda a+(i\lambda a)^2}{1+i\lambda a}+\frac{4}{5}e\cdot\frac{6+12i\lambda a+6\,(i\lambda a)^2+(i\lambda a)^3}{(1+i\lambda a)^2}\right].$$

If the spheroid make small oscillations of period $\dfrac{2\pi}{n}$, then

$$-\lambda^2\nu=in,$$

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* *Phil. Trans.*, 1880, Part I., p. 128.

and $$i\lambda = \sqrt{\frac{n}{2\nu}}(1+i) = n_1(1+i).$$

Thus $$\frac{3+3i\lambda a+(i\lambda a)^2}{1+i\lambda a} = \frac{3(1+n_1a)+in_1a(3+2n_1a)}{1+n_1a+in_1a}$$

$$= \frac{3+6n_1a+6n_1^2a^2+2n_1^3a^3+2in_1^2a^2(1+n_1a)}{1+2n_1a+2n_1^2a^2},$$

and $$\frac{6+12i\lambda a+6(i\lambda a)^2+(i\lambda a)^3}{(1+i\lambda a)^2}$$

$$= 2\,.\,\frac{3+6n_1a-n_1^3a^3+in_1a(6+6n_1a+n_1^2a^2)}{1+2n_1a+2in_1a(1+n_1a)}$$

$$= 2\,.\,\frac{3+12n_1a+24n_1^2a^2+23n_1^3a^3+12n_1^4a^4+2n_1^5a^5+in_1^3a^3(1+4n_1a+2n_1^2a^2)}{(1+2n_1a)^2+4n_1^2a^2(1+n_1a)^2}.$$

Collecting real and imaginary parts, the couple can be expressed in the form $$G = 2m\nu\,\omega_0\epsilon^{int}\,.\,\frac{b^2}{a^2}\left[K+inL+\epsilon(K'+inL')\right] \quad\ldots\ldots(27).$$

When $$\epsilon = 0,$$

the spheroid becomes a sphere, and

$$c = 0, \quad \alpha = \infty;$$

so that $$c\cosh\alpha = c\sinh\alpha = r;$$

and $$c\cosh\alpha_0 = a,$$

the radius of the sphere.

The solution in this case reduces to

$$w = \omega_0\epsilon^{-\lambda^2\nu t}\,.\,\frac{a}{r}\,.\,r\sin\theta\,.\,\frac{S_1(\lambda r)}{S_1(\lambda a)}$$

$$= \text{real part of } \omega_0\epsilon^{int}r\sin\theta\,.\left(\frac{a}{r}\right)^3.\,\frac{1+i\lambda r}{1+i\lambda a}\,.\,e^{i\lambda(a-r)}$$

$$= \text{real part of } \omega_0 r\sin\theta\,.\left(\frac{a}{r}\right)^3.\,\frac{1+n_1r+in_1r}{1+n_1a+in_1a}\,.\,e^{n_1(a-r)+i[nt+n_1(a-r)]}$$

$$= \omega_0 r\sin\theta\,.\left(\frac{a}{r}\right)^3.\,\frac{e^{n_1(a-r)}}{1+2n_1a+2n_1^2a^2}$$

$$\times\Big[\{(1+n_1r)(1+n_1a)+n_1^2ar\}\cos(nt+n_1\,.\,\overline{a-r})$$

$$-n_1(r-a)\,.\sin(n_1t+n_1\,.\,\overline{a-r})\Big] \quad\ldots\ldots\ldots(28).$$

And the couple on the sphere is

$$G = \tfrac{8}{3}\pi\rho\nu\,\omega_0\epsilon^{int}\,.\,a^3\,.\,\frac{3+3i\lambda a+(i\lambda a)^2}{1+i\lambda a} = 2m\nu\,\omega_0\epsilon^{int}\,(K+inL)\ldots(29).$$

7. Writing $\omega_0\,\epsilon^{int} = \dfrac{d\theta}{dt}$, so that θ is the angle turned through from the position of equilibrium, we have

$$G = 2m\nu\,.\,\frac{\mathrm{b}^2}{a^2}\left[K+\epsilon K'+(L+\epsilon L')\,\frac{d}{dt}\right]\frac{d\theta}{dt}$$

$$= \left[K_1+\epsilon K_1'+(L_1+\epsilon L_1')\,\frac{d}{dt}\right]\frac{d\theta}{dt}.$$

And the equation of motion of the spheroid is

$$I\ddot{\theta} = -G-In_0^2\,\theta,$$

or

$$(I+L_1+\epsilon L_1')\,\ddot{\theta}+(K_1+\epsilon K_1')\,\dot{\theta}+In_0^2\,\theta = 0,$$

where I is the moment of inertia of the spheroid, and n_0 is what n becomes when the fluid is absent.

The solution of this is

$$\theta = A\epsilon^{-\delta t}\sin\,(nt+a),$$

where δ, the modulus of decay, is equal to

$$\frac{1}{2}\cdot\frac{K_1+\epsilon K_1'}{I+L_1+\epsilon L_1'} = \frac{\nu K/2a^2}{M/5m+nL/2n_1^2a^2}\left\{1+\epsilon\,.\frac{K'}{K}-\epsilon\,.\,\frac{nL'/2n_1^2a^2}{M/5m+nL/2n_1^2a^2}\right\}.$$

If the viscosity be small, n_1 is a fairly large quantity, and $nL'/2n_1^2a^2$ is small. Hence, since this is multiplied by ϵ, it might be neglected. Also, $\nu/2a^2 = n/4n_1^2a^2$. Hence

$$\delta = \frac{1}{2}\cdot\frac{n\,(3+n_1a)}{M\,(1+2n_1a+2n_1^2a^2)/5m+(1+n_1a)}$$

$$\times\left\{1+\frac{8}{25}\epsilon\,.\,\frac{23+12n_1a+2n_1^2a^2}{32+20n_1a+4n_1^2a^2}\right\}\text{ (approximately)},$$

and the ratio of the increase of moment of inertia to the moment of inertia about the axis is

$$\frac{\delta I}{I} = I^{-1}(L_1+\epsilon L_1') = \frac{5mn}{M}\,(L+\epsilon L')\,/\,2n_1^2a^2$$

$$= \frac{5m}{M}\left[\frac{1+n_1a}{1+2n_1a+2n_1^2a^2}+\frac{2}{25}\epsilon\,.\,\frac{2+n_1a}{2+2n_1a+n_1^2a^2}\right]\text{ (nearly).}$$

Taking ν, for water in C.G.S. units, $= \cdot013$, σ, the density of the spheroid (for brass), $= 8$, we get the following results :—

τ	a	$\frac{\delta I}{I}$ = ratio of increase of moment of inertia.	δ^{-1} = time in which vibration falls to e^{-1} of its initial value.
10 secs.	1 c.m.	$\cdot0631+\cdot0129\epsilon$	$37\cdot22-7\cdot4610\epsilon$ secs.
	2 „	$\cdot0321+\cdot0071\epsilon$	$85\cdot10-15\cdot517\epsilon$
	3 „	$\cdot0214+\cdot0047\epsilon$	$134\cdot08-23\cdot399\epsilon$
	4 „	$\cdot0163+\cdot0034\epsilon$	$183\cdot88-31\cdot681\epsilon$
20 secs.	2 „	$\cdot0452+\cdot0051\epsilon$	$113\cdot44-22\cdot079\epsilon$
	4 „	$\cdot0227+\cdot0024\epsilon$	$257\cdot4\ \ -44\cdot543\epsilon$

In these approximations the assumption we have made is that terms of the form $\mu_2\lambda^4c^4$ may be neglected. Now $\lambda^2, \propto n\nu^{-1}$, is a fairly large quantity ; *e.g.*, if

$$n = 10, \quad \lambda \text{ or } n_1 = 4\cdot9,$$

but the numerical factor μ_2 is small. Thus the assumption is equivalent to taking ϵ, the ellipticity, small.

It is worthy of notice, too, that $K'/2n_1^2a^2$ and $nL'/2n_1^2a^2$ are small quantities, and these were the real and imaginary parts of the factor $\frac{5S_1S_2+\lambda a\,(S_1^2+S_2^2)}{\lambda^2a^2S_1^2}$, by which the small terms in (26) were multiplied ; hence the error in neglecting the small terms of the second order was probably not large. We see that the tendency is to produce a very small increase in the moment of inertia ; and an increase in δ ; so that the oscillations damp more quickly than for a sphere.

8. The preceding results are applicable, with slight alterations, to the case of a planetary spheroid. For, writing

$$\varpi = c\cosh a \sin\beta, \quad z = c\sinh a\cos\beta,$$

so that β is still measured from the axis of the spheroid, the equation of motion of the spheroid becomes

$$\frac{d^2w}{da^2}+\frac{d^2w}{d\beta^2}+\tanh a\,\frac{dw}{da}+\cot\beta\,\frac{dw}{d\beta}-\left(\frac{1}{\sin^2\beta}-\frac{1}{\cosh^2 a}\right)w$$

$$=\frac{c^2}{\nu}\left(\sinh^2 a+\cos^2\beta\right)\frac{dw}{dt},$$

and the surface conditions are

$$w = c \cosh \alpha_0 \sin \beta \,.\, \omega_0 \epsilon^{-\lambda^2 \nu t},$$

or

$$B \left[w - c \cosh \alpha_0 \sin \beta \, \omega_0 \, \epsilon^{-\lambda^2 \nu t} \right] = p_{\alpha\phi}.$$

Assuming, as before, that $w \propto \epsilon^{-\lambda^2 \nu t}$, we may put

$$w = \Sigma \epsilon^{-\lambda^2 \nu t} \,.\, f(\alpha) \,.\, F(\beta),$$

where

$$\frac{d^2 F}{d\beta^2} + \cot \beta \; \frac{dF}{d\beta} - \frac{F}{\sin^2 \beta} = -(k + \lambda^2 c^2 \cos^2 \beta) \, F,$$

$$\frac{d^2 f}{d\alpha^2} + \tanh \alpha \, \frac{df}{d\alpha} + \frac{f}{\sinh^2 \alpha} = -(\lambda^2 c^2 \cosh^2 \alpha - k) f.$$

The first of these equations is the same as (7), except that the sign of $\lambda^2 c^2$ is altered.

If in the second we put $\xi = \lambda c \sinh \alpha$, $c \sinh \alpha$ being the length of the polar semi-axis of the confocal α, the equation reduces to

$$(\xi^2 + \lambda^2 c^2) \frac{d^2 f}{d\xi^2} + 2\xi \frac{df}{d\xi} + \frac{\lambda^2 c^2}{\xi^2 + \lambda^2 c^2} f = (k - \xi^2) f.$$

This is the same as (8), except that $-\lambda^2 c^2$ is written for $\lambda^2 c^2$; hence the solution will be the same function of ξ as our former solution was of η, the only difference being that in the equations which determine the coefficients a_0, a_1, ... a_n, the sign of $\lambda^2 c^2$ is altered throughout.

When the ellipticity is small, the solution is of the same form as before; but the sign of ϵ is changed. The effect is therefore to produce a very slight diminution in the moment of inertia about the axis, and to decrease the modulus of decay, so that the vibrations do not damp so quickly as for a sphere.

9. When the motion is steady, equation (2) becomes, on writing ζ for $\cosh \alpha$ and μ for $\cos \beta$,

$$(\zeta^2 - 1) \frac{d^2 w}{d\zeta^2} + 2\zeta \frac{dw}{d\zeta} - \frac{w}{\zeta^2 - 1} + (1 - \mu^2) \frac{d^2 w}{d\mu^2} - 2\mu \frac{dw}{d\mu} - \frac{w}{1 - \mu^2} = 0$$

If we put $\quad a^2 + \lambda = c^2 \zeta^2, \quad b^2 + \lambda = c^2 (\zeta^2 - 1),$

and suppose the fluid contained between two confocal spheroids which rotate with angular velocities ω_0, ω_1, the solution is easily found to be

$$w = c\varpi \left[\omega_1 (A - A_0) + \omega_0 (A_1 - A) \right] / (A_1 - A_0) \ldots\ldots\ldots\ldots (30),$$

where $$A = \int_0^\lambda \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^2},$$

and A_1, A_0 are the values of A at the outer and inner spheroids.*

The couple resisting the motion of the spheroid is

$$G = -\iint p_{s\phi}\,.\,dS\,.\,\varpi = -\frac{16}{3}\pi\mu\,.\,\frac{\omega_1-\omega_0}{A_1-A_0}\;\ldots\ldots\ldots\ldots(31).$$

The corresponding results for planetary spheroids are got by writing $$A = \int_0^\lambda \frac{d\lambda}{(a^2+\lambda)^2(b^2+\lambda)^{\frac{1}{2}}}.$$

When the rotating spheroid degenerates into a disc, ζ or $\sinh a$, for the inner spheroid, $= 0$. Hence

$$w = \varpi\left[\omega_1\left(\tan^{-1}\zeta+\frac{\zeta}{\zeta^2+1}\right)+\omega_0\left(\cot^{-1}\zeta-\frac{\zeta}{\zeta^2+1}\right)_{\zeta_1}^{\zeta}\right]\Big/\left(\tan^{-1}\zeta_1+\frac{\zeta_1}{\zeta_1^2+1}\right).$$

And the couple resisting the motion is

$$G = \frac{16}{3}\pi\mu c^3\,\frac{\omega_1-\omega_0}{\tan^{-1}\zeta_1+\dfrac{\zeta_1}{\zeta_1^2+1}}.$$

If the fluid be unlimited externally $\omega_1 = 0$, $\zeta_1 = \infty$, so that

$$\left.\begin{aligned}\frac{\pi}{2}\,w &= \varpi\omega_0\left(\cot^{-1}\zeta-\frac{\zeta}{\zeta^2+1}\right)\\ \text{and}\qquad G &= -\frac{32}{3}\,\mu\omega_0 c^3\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

II. *Oscillations of a Spheroid along its axis.*

10. Suppose the spheroid suspended, with its axis horizontal, by a fine fibre, whose length is great compared with the dimensions of the spheroid. Then, neglecting the effects of the fibre on the fluid, we may regard the centre of the spheroid as performing small oscillations along its axis; while, since the dimensions of the spheroid are small compared with the length of the pendulum, we may neglect the effects of rotation.

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* *Cf.*, Basset, *Hydromechanics*, Chap. xxi., p. 284, ex. 8.

If the motion start from rest, it will be symmetrical about the axis at any subsequent time. Denoting by u', v' the velocities in the directions in which ϖ and z increase, we have

$$u' = -\frac{1}{\varpi}\frac{d\psi}{dz}, \quad v' = \frac{1}{\varpi}\frac{d\psi}{d\varpi},$$

where ψ is Stokes' current function.

Neglecting squares and products of velocities, the equations of motion in cylindrical coordinates* are

$$\left.\begin{aligned} \frac{dQ}{dz} &= \frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{d}{dt}-\nu D\right)\psi, \\ -\frac{dQ}{d\varpi} &= \frac{1}{\varpi}\frac{d}{dz}\left(\frac{d}{dt}-\nu D\right)\psi, \end{aligned}\right\}$$

where $$D = \left(\frac{d}{dz}\right)^2 + \left(\frac{d}{d\varpi}\right)^2 - \frac{1}{\varpi}\frac{d}{d\varpi},$$

and $$Q = -p/\rho + g\varpi,$$

if the axis of z be taken along the axis of the spheroid, which is horizontal.

To our order of approximation these equations are the same,† if we take the origin at the centre of the spheroid, and moveable with it.

Writing $\varpi = c \sinh a \sin\beta$, $z = c\cosh a\cos\beta$,

and taking as our standard case that of a prolate spheroid, we have

$$\left.\begin{aligned} &\left(\cosh a \sin\beta\frac{d}{d\varpi} + \sinh a\cos\beta\frac{d}{dz}\right)Q \\ &= \frac{1}{c\sinh a\sin\beta}\left(\sinh a\cos\beta\frac{d}{d\varpi} - \cosh a\sin\beta\frac{d}{dz}\right)\left(\frac{d}{dt}-\nu D\right)\psi, \\ -&\left(\sinh a\cos\beta\frac{d}{d\varpi} - \cosh a\sin\beta\frac{d}{dz}\right)Q \\ &= \frac{1}{c\sinh a\sin\beta}\left(\cosh a\sin\beta\frac{d}{d\varpi} + \sinh a\cos\beta\frac{d}{dz}\right)\left(\frac{d}{dt}-\nu D\right)\psi, \end{aligned}\right\}$$

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* *Cf.* Basset, *Hydrodynamics*, Vol. II., p. 262.

† *Cf.* Stokes, *Camb. Phil. Trans.*, Vol. IV., Pt. II., p. 51.

or

$$\left.\begin{aligned} \frac{dQ}{d\alpha} &= \frac{1}{c\sinh\alpha\sin\beta}\,\frac{d}{d\beta}\left(\frac{d}{dt}-\nu D\right)\psi, \\ -\frac{dQ}{d\beta} &= \frac{1}{c\sinh\alpha\sin\beta}\,\frac{d}{d\alpha}\left(\frac{d}{dt}-\nu D\right)\psi, \end{aligned}\right\}$$

where
$$D = \frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}-\coth\alpha\frac{d}{d\alpha}-\cot\beta\frac{d}{d\beta}.$$

Eliminating Q, we find

$$\left(D-\frac{1}{\nu}\frac{d}{dt}\right)D\psi = 0.$$

The solution of this is $\psi = \psi_1+\psi_2$,

where ψ_1, ψ_2 satisfy the equations

$$\left.\begin{aligned} D\psi_1 &= 0, \\ \left(D-\frac{1}{\nu}\frac{d}{dt}\right)\psi_2 &= 0. \end{aligned}\right\}$$

Assuming that the time enters in the form of the factor $e^{-\lambda^2\nu t}$, these reduce to

$$D\psi_1 = 0,\quad (D-\lambda^2)\,\psi_2 = 0,$$

or

$$\left.\begin{aligned} \left[\frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}-\coth\alpha\frac{d}{d\alpha}-\cot\beta\frac{d}{d\beta}\right]\psi_1 &= 0 \\ \left[\frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}-\coth\alpha\frac{d}{d\alpha}-\cot\beta\frac{d}{d\beta}+\lambda^2c^2(\cosh^2\alpha-\cos^2\beta)\right]\psi_2 &= 0 \end{aligned}\right\} \quad \text{.................(33).}$$

The pressure at any point of the fluid is given by

$$-dQ = \frac{1}{c\sinh\alpha\sin\beta}\,\frac{d}{dt}\left(\frac{d\psi_1}{d\alpha}\,d\beta-\frac{d\psi_1}{d\beta}\,d\alpha\right) \quad \text{.........(34).}$$

11. To find the surface conditions, we must resolve the velocity of the spheroid along its axis, in directions normal and tangential to the surface.

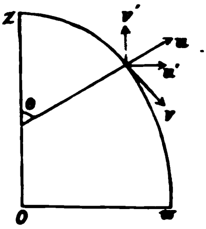
Let θ be the inclination of the normal at P to the axis of z, and p the perpendicular from the centre on the tangent plane at P.

Then

$$\frac{1}{p^2}=\frac{\varpi^2}{c^4\sinh^4\alpha_0}+\frac{z^2}{c^4\cosh^4\alpha_0}=\frac{1}{c^2}\,\frac{\cosh^2\alpha_0-\cos^2\beta}{\sinh^2\alpha_0\cosh^2\alpha_0}=\frac{h^{-2}}{c^4\sinh^2\alpha_0\cosh^2\alpha_0},$$

$$\cos\theta=\frac{pz}{c^2\cosh^2\alpha_0}=hc\sinh\alpha_0\cos\beta,$$

$$\sin\theta=\frac{p\varpi}{c^2\sinh^2\alpha_0}=hc\cosh\alpha_0\sin\beta;$$



therefore*

$$u=\text{velocity along the normal to }\alpha$$

$$=u'\sin\theta+v'\cos\theta=\frac{1}{\varpi}\left(\cos\theta\frac{d\psi}{d\varpi}-\sin\theta\frac{d\psi}{dz}\right)$$

$$=\frac{h}{\varpi}\,\frac{d\psi}{d\beta}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Also, $v=$ velocity along the normal to β

$$=u'\cos\theta-v'\sin\theta=-\frac{1}{\varpi}\left(\cos\theta\frac{d\psi}{dz}+\sin\theta\frac{d\psi}{d\varpi}\right)$$

$$=-\frac{h}{\varpi}\,\frac{d\psi}{d\alpha}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

If the spheroid be moving along its axis with velocity $Ve^{-\lambda^2\nu t}$, and the fluid stick, the boundary conditions are

$$u=\;Ve^{-\lambda^2\nu t}\cos\theta=\;Ve^{-\lambda^2\nu t}\,.\,hc\sinh\alpha_0\cos\beta,$$

$$v=-Ve^{-\lambda^2\nu t}\sin\theta=-Ve^{-\lambda^2\nu t}\,.\,hc\cosh\alpha_0\sin\beta,$$

or,

$$\left.\begin{aligned}\frac{d\psi}{d\alpha}&=Ve^{-\lambda^2\nu t}c^2\sinh\alpha_0\cosh\alpha_0\sin^2\beta\\ \frac{d\psi}{d\beta}&=Ve^{-\lambda^2\nu t}c^2\sinh^2\alpha_0\sin\beta\cos\beta\end{aligned}\right\}\;\ldots\ldots\ldots\ldots(37),$$

at the surface.

If slipping take place at the surface of the spheroid, we must substitute for the first of these the condition

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* These expressions might be written down without going through the transformation; for $2\pi\psi$ = flux across any area bounded by a stream line. Hence, if P be a point on a stream line, P' a point on a consecutive stream line in the plane passing through P, and the axis, $2\pi d\psi = 2\pi\varpi\,.\,PP'$ (velocity normal to PP'). Then by taking PP' perpendicular first to α and then to β, we get the above expressions.

$$\frac{Bh}{c \sinh \alpha_0 \sin \beta}\left(\frac{d\psi}{d\alpha} - Ve^{-\lambda^2\nu t} . c^3 \sinh \alpha_0 \cosh \alpha_0 \sin^2 \beta\right) = p_{\alpha\beta} \text{...(37}a\text{)}.$$

If the fluid extend to infinity, we have the additional condition that the velocity vanishes there.

12. Taking the equation $D\psi_1 = 0$,

and putting $\zeta = \cosh \alpha, \quad \mu = \cos\beta,$

we can write it in the form

$$(\zeta^2-1)\frac{d^2\psi_1}{d\zeta^2} + (1-\mu^2)\frac{d^2\psi_1}{d\mu^2} = 0.$$

The solution of this may be written

$$\psi_1 = f_1(\zeta)\, F_1(\mu),$$

where

$$\left.\begin{aligned} (\zeta^2-1)\frac{d^2f_1}{d\zeta^2} - n(n+1)f_1 = 0, \\ (1-\mu^2)\frac{d^2F_1}{d\mu^2} + n(n+1)F_1 = 0. \end{aligned}\right\}.$$

We want a solution which vanishes at infinity, hence we write

$$\psi_1 = \Sigma A_n (1-\mu^2)\frac{dP_n}{d\mu}(\zeta^2-1)\frac{dQ_n}{d\zeta} = \Sigma A_n \sin\beta \sinh\alpha \,.\, P_n^1(\mu) \,.\, Q_n^1(\zeta) \qquad \text{..................(38)},$$

where $P_n^1(\mu) = (1-\mu^2)^{\frac{1}{2}}\frac{dP_n}{d\mu}, \quad Q_n^1(\zeta) = (\zeta^2-1)^{\frac{1}{2}}\frac{dQ_n}{d\zeta}.$

The solution of the equation

$$(D-\lambda^2)\,\psi_2 = 0$$

may be written $\psi_2 = f_2(\alpha) \,.\, F_2(\beta),$

where F_2, f_2 satisfy the equations

$$\left.\begin{aligned} \frac{d^2F_2}{d\beta^2} - \cot\beta\frac{dF_2}{d\beta} = (\lambda^2c^2\cos^2\beta - k)\,F_2, \\ \frac{d^2f_2}{d\alpha^2} - \coth\alpha\frac{df_2}{d\alpha} = (k - \lambda^2c^2\cosh^2\alpha)\,f_2. \end{aligned}\right\}$$

Writing η for $\lambda c \cosh a$, and μ for $\cos\beta$, these become

$$(1-\mu^2)\frac{d^2F_2}{d\mu^2} = (\lambda^2c^2\mu^2-k)\,F_2 \quad \dots\dots\dots\dots\dots\dots (39),$$

$$(\eta^2-\lambda^2c^2)\frac{d^2f_2}{d\eta^2} = (k-\eta^2)\,f_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots (40).$$

Now, since

$$\left[(1-\mu^2)\left(\frac{d}{d\mu}\right)^2+k\right](1-\mu^2)^{\frac{1}{2}}P_n^1 = -\left[n\,(n+1)-k\right](1-\mu^2)^{\frac{1}{2}}P_n^1,$$

and $\mu^2P_n^1$ can be expressed in terms of P_{n+2}^1, P_n^1, P_{n-2}^1 by equation (9), equation (39) can be satisfied by the expression

$$\vartheta(\mu) = C\sin\beta\left[a_0P_1^1-\frac{2}{3.5}\,a_1P_3^1+\dots\right.$$
$$\left.\dots+(-1)^r\frac{1.2\,\dots\,(n-1)}{1.3\,\dots\,(2n-1)}\,a_rP_n^1+\dots\right],$$

where $$n = 2r+1,$$

and a_0, a_1, ... are determined by the same set of equations as before.

Writing $$f_2(\eta) = (\eta^2-\lambda^2c^2)^{\frac{1}{2}}f(\eta),$$

equation (40) reduces to

$$(\eta^2-\lambda^2c^2)\frac{d^2f}{d\eta^2}+2\eta\frac{df}{d\eta}-\frac{\lambda^2c^2}{\eta^2-\lambda^2c^2}f = (k-\eta^2)\,f.$$

This is the same as equation (8), and the solution is

$$\Omega(\eta) = \frac{(\eta^2-\lambda^2c^2)^{\frac{1}{2}}}{\eta}\left[2a_0S_1+\tfrac{8}{5}a_1S_3+\dots+\frac{1.2\,\dots\,(n+1)}{1.3\,\dots\,(2n-1)}\,a_rS_n+\dots\right],$$

where, as before, $$n = 2r+1.$$

Hence

$$\psi_2 = \Sigma_{r=0}^{r=\infty}\,e^{-\lambda^2\nu t}B_r\sin\beta\sinh a\,\Omega_r(\eta)\,\vartheta_r(\mu)$$

$$= \Sigma_{r=0}^{r=\infty}\,e^{-\lambda^2\nu t}(-1)^r\frac{1.2\,\dots\,(n-1)}{1.3\,\dots\,(2n-1)}\sin\beta\sinh a\tanh a$$
$$\times\left[2b_{0r}S_1+\tfrac{8}{5}b_{1r}S_3+\dots+\dots\right]P_n^1$$

$$= \Sigma_{r=0}^{r=\infty}\,\Sigma_{s=0}^{s=\infty}\,e^{-\lambda^2\nu t}(-1)^r\frac{2^n.\,n!\;\overline{n-1}!}{\overline{2n}!}\cdot\frac{2^m.\,m!\;\overline{m+1}!}{\overline{2m}!}\,b_{sr}$$
$$\times\sin\beta\sinh a\tanh a\,S_m(\eta)\,P_n^1(\mu) \quad \dots\dots (41),$$

where $$n = 2r+1, \quad m = 2s+1,$$

and $$b_{sr} = B_0 a_{0s} a_{0r} + B_1 a_{1s} a_{1r} + \ldots + B_r a_{rs} a_{rr} + \ldots.$$

We might obtain the value of ψ_2 in terms of associated functions of the second kind. For if, instead of the substitution η for $\lambda c \cosh \alpha$, we put ζ for $\cosh \alpha$, equation (40) becomes

$$(\zeta^2-1)\frac{d^2 f_2}{d\zeta^2} = (k-\lambda^2 c^2 \zeta^2) f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40a).$$

If we write $$f_2(\zeta) = (\zeta^2-1)^{\frac{1}{2}} . f(\zeta),$$

this becomes $$(\zeta^2-1)\frac{d^2 f}{d\zeta^2} + 2\zeta\frac{df}{d\zeta} - \frac{1}{\zeta^2-1}f = (k-\lambda^2 c^2 \zeta^2) f.$$

Hence, in terms of associated functions of the second kind,

$$f_2(\zeta) = C' \sinh \alpha \Big[a_0 Q_1^1 - \frac{2}{3.5} a_1 Q_3^1 + \ldots$$
$$\ldots + (-1)^r a_r \frac{1.2\ldots(n-1)}{1.3\ldots(2n-1)} Q_n^1 + \ldots \Big],$$

and the corresponding value of ψ_2 is

$$\psi_2 = \Sigma_{r=0}^{r=\infty} \epsilon^{-\lambda^2 \nu t} (-1)^r \frac{1.2\ldots(n-1)}{1.3\ldots(2n-1)} \sin\beta \sinh\alpha$$
$$\times \Big[\mathrm{b}_{0r} Q_1^1 - \frac{2}{3.5} \mathrm{b}_{1r} Q_3^1 + \ldots \Big] P_n^1$$

$$= \Sigma_{r=0}^{r=\infty} \Sigma_{s=0}^{s=\infty} \epsilon^{-\lambda^2 \nu t} (-1)^{r+s} \frac{2^{m+n} . n!\, n-1!\, m!\, m-1!}{2n!\, 2m!} \mathrm{b}_{sr}$$
$$\times \sin\beta \sinh\alpha\, Q_m^1(\zeta)\, P_n^1(\mu) \quad \ldots\ldots (41a),$$

where $$n = 2r+1, \quad m = 2s+1.$$

13. The arbitrary constants are determined from the surface conditions. At the surface we have, from (37),

$$\frac{d\psi}{d\mu} = -V\epsilon^{-\lambda^2 \nu t} c^2 \sinh^2 \alpha_0 \cos\beta = -V\epsilon^{-\lambda^2 \nu t} c^2 \sinh^2 \alpha_0 . P_1(\mu).$$

Now, since P_n, Q_n are ordinary zonal harmonics of the first and second kinds,

$$\frac{d}{d\mu}\left(\overline{1-\mu^2}\,\frac{dP_n}{d\mu}\right) = -n\,(n+1)\,P_n,$$

$$\frac{d}{d\zeta}\left(\overline{\zeta^2-1}\,\frac{dQ_n}{d\zeta}\right) = n\,(n+1)\,Q_n,$$

and the surface condition becomes

$$Vc^2\sinh^2 a_0\,.\,P_1 = \Sigma_0^\infty A_r\,(\zeta_0^2-1)\,\frac{dQ_n}{d\zeta_0}\,.\,n\,(n+1)\,P_n$$

$$+\Sigma_{r=0}^{r=\infty}\Sigma_{s=0}^{s=\infty}(-1)^r\,.\,\frac{2^{m+n}\,.\,n!\,n+1!\,m!\,m+1!}{2m!\,2n!}\,\mathrm{b}_{sr}\sinh a_0\tanh a_0 S_m(\eta_0)\,.\,P_n.$$

Hence, equating coefficients of P_1, P_2, ..., we get

$$A_r Q_1^1(\zeta_0)+(-1)^r\,\Sigma_{s=0}^{s=\infty}\frac{2^{m+n}\,.\,n!\,n-1!\,m!\,m+1!}{2m!\,2n!}\,\mathrm{b}_{sr}\tanh a_0 S_m(\eta_0)$$

$$= \tfrac{1}{2}Vc^2\sinh a_0 \text{ or } 0 \text{.......................(42),}$$

according as r is or is not equal to zero, m being equal, as before, to $2s+1$, and n to $2r+1$.

If there be no slipping at the surface, the other condition is

$$\frac{d\psi}{d\zeta_0} = Ve^{-\lambda^2\kappa}\,.\,c^2\cosh a_0\sin\beta\,.\,P_1^1(\mu).$$

Now
$$e^{\lambda^2\kappa}\,\frac{d\psi}{d\zeta} = \Sigma A_r\,.\,n\,(n+1)\,Q_n(\zeta_0)\,.\,(1-\mu^2)^{\frac{1}{2}}P_n^1(\mu)$$

$$+\Sigma_{r=0}^{r=\infty}\Sigma_{s=0}^{s=\infty}(-1)^r\,.\,\frac{2^{m+n}\,.\,n!\,n-1!\,m!\,m+1!}{2n!\,2m!}$$

$$\times\frac{d}{d\zeta_0}\left[\sinh a_0\tanh a_0 S_m\right].\,(1-\mu^2)^{\frac{1}{2}}P_n^1.$$

And
$$\frac{d}{d\zeta}\left[\sinh a\tanh a\,S_m\right]$$

$$= \frac{d}{d\eta}\left(\frac{\eta^2-\lambda^2c^2}{\eta}\,.\,S_m\right) = \left[(m+1)-(m-1)\frac{\lambda^2c^2}{\eta^2}\right]S_m + \frac{\eta^2-\lambda^2c^2}{\eta}\,.\,S_{m+1}.$$

Hence, equating coefficients,

$$n\,(n+1)\,A_r\,Q_n(\zeta_0)+(-1)^r\,\Sigma_{s=0}^{s=\infty}\frac{2^{m+n}\,.\,n!\,n-1!\,m!\,m+1!}{2m!\,2n!}\,\mathrm{b}_{sr}$$

$$\times\left[\left\{(m+1)-(m-1)\frac{\lambda^2c^2}{\eta_0^2}\right\}S_m+\frac{\eta_0^2-\lambda^2c^2}{\eta_0}S_{m+1}\right] = Vc^2\cosh a_0 \text{ or } 0$$

$$\text{............(43),}$$

according as r is or is not equal to zero; m being equal to $2s+1$, and n to $2r+1$.

If we take the second form of ψ_2, we get

$$e^{\lambda^2\nu t}\,\frac{d\psi}{d\zeta} = \Sigma A_r\, n\,(n+1)\, Q_n(\zeta_0)\,(1-\mu^2)^{\frac{1}{2}} P_n^1(\mu)$$

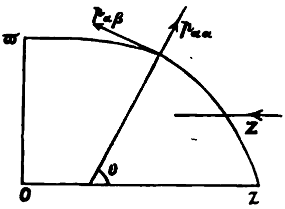
$$+\Sigma_{r=0}^{r=\infty}\Sigma_{s=0}^{s=\infty}(-1)^{r+s}\,.\,\frac{2^{m+n}\,.\,n!\;\overline{n-1}!\;m!\;\overline{m+1}!}{2m!\;2n!}\,.\,\mathrm{b}_{sr}\,Q_m(\zeta_0)(1-\mu^2)^{\frac{1}{2}}P_n^1(\mu),$$

$$\therefore\quad n(n+1)A_rQ_n(\zeta_0)+(-1)^r\Sigma_{s=0}^{s=\infty}(-1)^s\,.\,\frac{2^{m+n}\,.\,n!\;\overline{n-1}!\;m!\;\overline{m+1}!}{2m!\;2n!}$$

$$\times\,\mathrm{b}_{sr}\,.\,Q_m(\zeta_0) = Vc^2\cosh\alpha_0 \text{ or } 0\text{..............}(43a),$$

according as r is or is not zero.

14. We now proceed to find the resistance exerted by the liquid on the spheroid. Let Z denote this resistance. Then, resolving all the stresses on the spheroid in the direction of the axis of z, and integrating over the surface of the spheroid, we have



$$Z=\iint(-p_{\alpha\alpha}\cos\theta+p_{\alpha\beta}\sin\theta)\,\varpi\,d\phi\,.\,h^{-1}\,d\beta.$$

At the surface of the spheroid, from (35) and (36),

$$u=\frac{h}{\varpi}\,.\,\frac{d\psi}{d\beta},\qquad v=-\frac{h}{\varpi}\,.\,\frac{d\psi}{d\beta};$$

therefore
$$p_{\alpha\beta}=\nu\rho\left[\frac{d}{d\alpha}(hv)+\frac{d}{d\beta}(hu)\right]$$

$$=\nu\rho\left[\frac{d}{d\beta}\left(\frac{h^2}{\varpi}\,.\,\frac{d\psi}{d\beta}\right)-\frac{d}{d\alpha}\left(\frac{h^2}{\varpi}\,.\,\frac{d\psi}{d\alpha}\right)\right]$$

$$=\frac{\nu\rho h^2}{\varpi}\left[\frac{d^2\psi}{d\beta^2}-\cot\beta\frac{d\psi}{d\beta}-\frac{d^2\psi}{d\alpha^2}+\coth\alpha\frac{d\psi}{d\alpha}\right.$$

$$\left.-2c^2h^2\left(\sin\beta\cos\beta\frac{d\psi}{d\beta}-\sinh\alpha\cosh\alpha\frac{d\psi}{d\alpha}\right)\right],$$

since
$$\frac{dh}{d\alpha}=-h^3c^2\sinh\alpha\cosh\alpha,\qquad\frac{dh}{d\beta}=-h^3c^2\sin\beta\cos\beta,$$

$$\frac{d\varpi}{d\alpha}=c\cosh\alpha\sin\beta,\qquad\frac{d\varpi}{d\beta}=c\sinh\alpha\cos\beta.$$

Making use of equations (33), we find

$$-\left(\frac{d^2\psi}{d\alpha^2}-\coth\alpha\frac{d\psi}{d\alpha}\right)=\frac{d^2\psi}{d\beta^2}-\cot\beta\frac{d\psi}{d\beta}+\lambda^2c^2(\cosh^2\alpha-\cos^2\beta)\,\psi_1.$$

Also, at the surface,

$$\frac{d^2\psi}{d\beta^2} = Ve^{-\lambda^2\nu t} \,.\, c^2 \sinh^2 \alpha_0 \,(\cos^2\beta - \sin^2\beta)\,;$$

therefore $$\frac{d^2\psi}{d\beta^2} - \cot\beta\,\frac{d\psi}{d\beta} = -Ve^{-\lambda^2\nu t}\, c^2 \sinh^2 \alpha_0 \sin^2\beta.$$

And the surface value of

$$-2c^2k^2\left(\sin\beta\cos\beta\,\frac{d\psi}{d\beta} - \sinh\alpha\cosh\alpha\,\frac{d\psi}{d\alpha}\right),$$

is $$2Ve^{-\lambda^2\nu t} \,.\, c^2 \sinh^2 \alpha_0 \sin^2\beta.$$

Hence, at the surface,

$$p_{\alpha\beta} = \frac{\nu\rho h^2}{\varpi}\left[2\left(\frac{d^2\psi}{d\beta^2} - \cot\beta\,\frac{d\psi}{d\beta}\right) - c^2k^2\left(\sin 2\beta\,\frac{d\psi}{d\beta} - \sinh 2\alpha\,\frac{d\psi}{d\alpha}\right) + \lambda^2 h^{-2}\psi_2\right]$$

$$= \frac{\lambda^2\nu\rho}{\varpi} \,.\, \psi_2 \quad\text{...} \quad (44).$$

Again, from (4),

$$p_{\alpha\alpha} = -p + \nu\rho h\left(\frac{du}{d\alpha} - \frac{v}{h}\,\frac{dh}{d\beta}\right).$$

But, at the surface,

$$\frac{du}{d\alpha} - \frac{v}{h}\,\frac{dh}{d\beta} = \frac{d}{d\alpha}\left(\frac{h}{\varpi}\,\frac{d\psi}{d\beta}\right) + \frac{1}{\varpi}\,\frac{dh}{d\beta}\cdot\frac{d\psi}{d\alpha}$$

$$= \frac{h}{\varpi}\left[\frac{d^2\psi}{d\alpha\, d\beta} - \coth\alpha\,\frac{d\psi}{d\beta} - \tfrac{1}{2}c^2h^2\left(\sinh 2\alpha\,\frac{d\psi}{d\beta} + \sin 2\beta\,\frac{d\psi}{d\alpha}\right)\right]_{\alpha=\alpha_0}.$$

And each of these expressions has the same surface value

$$Ve^{-\lambda^2\nu t}\, c^2 \sinh\alpha_0 \cosh\alpha_0 \sin\beta\cos\beta.$$

Hence, at the surface $$\frac{du}{d\alpha} - \frac{v}{h}\,\frac{dh}{d\beta} = 0,$$

and the surface value of $p_{\alpha\alpha}$ is $-p$ ....(45).

The force resisting the spheroid is therefore

$$Z = \iint (-p_{\alpha\alpha}\cos\theta + p_{\alpha\beta}\sin\theta)\,\varpi\, d\phi\, h^{-1}\, d\beta,$$

from $\phi = 0$ to 2π, $\beta = 0$ to π; therefore

$$Z = 2\pi\int_0^\pi p\cos\theta\,.\,\varpi h^{-1}\,d\beta + 2\pi\lambda^2\nu\rho\int_0^\pi \psi_2\sin\theta\,.\,h^{-1}\,d\beta.$$

Now

$$\int_0^\pi p\cos\theta\,.\,\varpi h^{-1}\,d\beta = \int_0^\pi p\,.\,hc\sinh\alpha_0\cos\beta\,.\,c\sinh\alpha_0\sin\beta\,.\,h^{-1}\,d\beta$$

$$= c^2\sinh^2\alpha_0\int_0^\pi p\sin\beta\cos\beta\,d\beta$$

$$= -\tfrac{1}{2}c^2\sinh^2\alpha_0\int_0^\pi \frac{dp}{d\beta}\sin^2\beta\,d\beta.$$

But, from (34),

$$-dQ = \frac{1}{c\sinh\alpha_0\sin\beta}\,\frac{d}{dt}\left(\frac{d\psi_1}{d\alpha}\,d\beta - \frac{d\psi_1}{d\beta}\,d\alpha\right),$$

where $Q = -p/\rho + gc\sinh\alpha\sin\beta$;

therefore $$-\frac{1}{\rho}\,\frac{dp}{d\beta} = \frac{dQ}{d\beta} - gc\sinh\alpha\cos\beta$$

$$= -\frac{1}{c\sinh\alpha\sin\beta}\,\frac{d^2\psi_1}{d\alpha\,dt} - gc\sinh\alpha\cos\beta.$$

And

$$\int_0^\pi p\cos\theta\,.\,\varpi h^{-1}\,d\beta = -\tfrac{1}{2}\rho c^2\sinh^2\alpha_0$$

$$\times\int_0^\pi\left[\frac{1}{c\sinh\alpha_0\sin\beta}\,\frac{d^2\psi_1}{d\alpha\,dt} + gc\sinh\alpha_0\cos\beta\right]\sin^2\beta\,d\beta$$

$$= -\tfrac{1}{2}\rho c\sinh\alpha_0\int_{-1}^{+1}\frac{d^2\psi_1}{d\alpha\,dt}\,d\mu = \tfrac{1}{2}\lambda^2\nu\rho c\sinh\alpha_0\int_{-1}^{+1}\frac{d\psi_1}{d\alpha}\,d\mu.$$

Also,

$$\int_0^\pi \psi_2\sin\theta\,.\,h^{-1}d\beta = \int_0^\pi \psi_2\,.\,c\cosh\alpha_0\,.\,\sin\beta\,d\beta = c\cosh\alpha_0\int_{-1}^{+1}\psi_2\,d\mu;$$

$$\therefore\quad Z = \pi\rho\lambda^2\nu\left[c\sinh\alpha_0\int_{-1}^{+1}\frac{d\psi_1}{d\alpha}\,d\mu + 2c\cosh\alpha_0\int_{-1}^{+1}\psi_2\,d\mu\right]\ldots\ldots(46).$$

This corresponds exactly in form with the known result* for a spherical pendulum.

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* *Cf.* Stokes, *Camb. Phil. Trans.*, Vol. IX., Part. ii., p. 31.

Since $$\psi_1 = \Sigma_{r=0}^{r=\infty} A_r \epsilon^{-\lambda^2 \nu t} (1-\mu^2) \frac{dP_n}{d\mu} (\zeta^2-1) \frac{dQ_n}{d\zeta},$$

where $$n = 2r+1,$$

and $$\int_{-1}^{+1} (1-\mu^2) \frac{dP_n}{d\mu}\, d\mu = \left[P_n (1-\mu^2)\right]_{-1}^{+1} + 2\int_{-1}^{+1} \mu P_n d\mu = \tfrac{4}{3} \text{ or } 0,$$

according as n is or is not equal to unity; the only term in $\frac{d\psi_1}{da}$ which does not disappear on integration is the first. And

$$\frac{d}{da}\left(\overline{\zeta^2-1}\, \frac{dQ_n}{d\zeta}\right) = 2 \sinh a \,.\, Q_1\,;$$

therefore $$\int_{-1}^{+1} \frac{d\psi_1}{da}\, d\mu = \tfrac{8}{3} A_0 \epsilon^{-\lambda^2 \nu t} \,.\, \sinh a_0 \,.\, Q_1(\zeta_0).$$

Similarly, the only term in ψ_2 which does not vanish is that containing $(1-\mu^2)^{\frac{1}{2}} P_1^1(\mu)$, viz.:—

$$\epsilon^{-\lambda^2 \nu t} \sin\beta \sinh a \left[2\mathrm{b}_{00} S_1 + \tfrac{8}{5}\mathrm{b}_{10} S_3 + \ldots\right] P_1^1(\mu).$$

Hence $$\begin{aligned} Z &= \tfrac{8}{3}\pi\rho\lambda^2 \nu \epsilon^{-\lambda^2 \nu t} \,.\, c \sinh a_0 \\ &\quad \times \left[A_0 \sinh a_0\, Q_1(\zeta_0) + \cosh a_0 \left(2\mathrm{b}_{00} S_1 + \tfrac{8}{5}\mathrm{b}_{10} S_3 + \ldots\right)\right] \\ &= \tfrac{8}{3}\pi\rho\lambda^2 \nu \epsilon^{-\lambda^2 \nu t}\, c \sinh a_0 \\ &\quad \times \left[A_0 \sinh a_0\, Q_1(\zeta_0) + \cosh a_0 \left\{\tfrac{1}{2} Vc^2 \sinh a_0 - A_0 Q_1^1(\zeta_0)\right\}\right], \end{aligned}$$

as we see on substituting from equation (42).

Now* $$Q_1(\zeta) = \zeta \coth^{-1}\zeta - 1,$$

$$Q_1^1(\zeta) = (\zeta^2-1)^{\frac{1}{2}} \frac{dQ_1}{d\zeta} = \left[(\zeta^2-1)^{\frac{1}{2}} \coth^{-1}\zeta - \frac{\zeta}{(\zeta^2-1)^{\frac{1}{2}}}\right];$$

therefore $$(\zeta^2-1)^{\frac{1}{2}} Q_1(\zeta) - \zeta\, Q_1^1(\zeta) = \operatorname{cosech} a.$$

Hence $$Z = \tfrac{8}{3}\pi\rho\lambda^2 \nu \epsilon^{-\lambda^2 \nu t} \,.\, c \left[A_0 + \tfrac{1}{2} Vc^2 \sinh^2 a_0 \cosh a_0\right].$$

15. Let us suppose the ellipticity of the spheroid is so small that terms involving $\lambda^4 c^4$ may be neglected. Then we have, approximately,

$$\tanh a = 1-\epsilon, \quad \operatorname{sech}^2 a = 2\epsilon,$$

$$Q_1(\zeta) = \frac{1}{3\zeta^2} + \frac{1}{5\zeta^4}, \quad Q_1^1(\zeta) = -\frac{2}{3} \,.\, \frac{(\zeta^2-1)^{\frac{1}{2}}}{\zeta^3} \left(1 + \frac{6}{5\zeta^2}\right).$$

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* *Cf.* Bryan, *Proc. Camb. Phil. Soc.*, Vol. VI.

To find Z the only coefficient we require is A_0. Remembering that A_0, B_0 differ from a constant by a term of the order λ^2c^2; A_1, B_1 are order λ^2c^3, and so on, we have

$$\tfrac{1}{2}Vc^2\cosh a_0 = A_0\coth a_0\, Q_1^1(\zeta_0)+2B_0\left(S_1+\tfrac{4}{5}a_{01}S_3\right),$$

$$\begin{aligned} Vc^2\cosh a_0 &= 2A_0Q_1(\zeta_0)+2B_0\left(2S_1+\frac{\eta_0^2-\lambda^2c^2}{\eta}S_2\right) \\ &\qquad +\tfrac{8}{5}B_0a_{01}\left(4S_3+\frac{\eta_0^2-\lambda^2c^2}{\eta_0}S_4\right) \\ &= 2A_0Q_1(\zeta_0)+2B_0\Big[2S_1+\lambda c\sinh a_0\tanh a_0 S_2 \\ &\qquad +\tfrac{4}{5}a_{01}\left(4S_3+\lambda aS_4\right)\Big]. \end{aligned}$$

Eliminating B_0, and writing for $Q_1(\zeta_0)$, $Q_1^1(\zeta_0)$ their approximate values in the small terms, we get

$$\begin{aligned} 2A_0\Big[S_1\left\{Q_1(\zeta_0)-\coth a_0\, Q_1^1(\zeta_0)\right\}&+\tfrac{1}{3}\lambda c\sinh a_0\, Q_1^1(\zeta_0)\, S_2 \\ &+\tfrac{4}{3}a_{01}\operatorname{sech}^2 a_0\left(S_3+\tfrac{1}{5}\lambda aS_4\right)\Big] \\ = -Vc^2\cosh a_0\Big[\tfrac{1}{2}\lambda c\sinh a_0\tanh a_0 S_2&+\tfrac{4}{5}a_{01}\left(S_3+\tfrac{1}{2}\lambda aS_4\right)\Big], \end{aligned}$$

or
$$\begin{aligned} A_0\Big[S_1+\tfrac{1}{3}\lambda a\left(1-\tfrac{8}{5}\epsilon\right)S_2+\tfrac{4}{3}a_{01}\left(S_3+\tfrac{1}{5}\lambda aS_4\right)\Big] \\ = -\tfrac{1}{2}V\frac{ab^2}{c}\Big[\tfrac{1}{2}\lambda a\left(1-2\epsilon\right)S_2+\tfrac{4}{5}a_{01}\left(S_3+\lambda aS_4\right)\Big], \end{aligned}$$

where a, b are the semi-axes of the principal elliptic section.

Hence

$$\begin{aligned} Z &= \tfrac{8}{3}\pi\rho\lambda^2\nu\,.\,\epsilon^{-\lambda^2\nu t}.c\left[A_0+\tfrac{1}{2}Vc^2\sinh^2 a_0\cosh a_0\right] \\ &= m\lambda^2\nu\,.\,V\epsilon^{-\lambda^2\nu t}\,\frac{S_1-\frac{1}{6}\lambda a\left(1-\frac{14}{5}\epsilon\right)S_2+\frac{4}{15}a_{01}\left(2S_3-\frac{1}{2}\lambda aS_4\right)}{S_1+\frac{1}{3}\lambda a\left(1-\frac{8}{5}\epsilon\right)S_2+\frac{4}{3}a_{01}\left(S_3+\frac{1}{5}\lambda aS_4\right)} \\ &= m\lambda^2\nu\,.\,V\epsilon^{-\lambda^2\nu t}\,\frac{S_1-\frac{1}{6}\lambda aS_2+\frac{1}{75}\lambda a\epsilon\left\{35S_2+8\lambda aS_3-2\lambda^2a^2S_4\right\}}{S_1+\frac{1}{3}\lambda aS_2-\frac{4}{75}\lambda a\epsilon\left\{10S_2-5\lambda aS_3-\lambda^2a^2S_4\right\}}. \end{aligned}$$

Writing, for shortness, x for $i\lambda a$, this reduces to

$$\begin{aligned} Z &= -\tfrac{1}{2}m\lambda^2\nu\,.\,V\epsilon^{-\lambda^2\nu t}\,\frac{x^2+9x+9+\frac{2}{25}\epsilon\left(2x^4+28x^3+103x^2+225x+225\right)}{x^2+\frac{4}{25}\epsilon\left(x^4+5x^3+5x^2\right)} \\ &= -\tfrac{1}{2}m\lambda^2\nu\,.\,V\epsilon^{-\lambda^2\nu t}\left[1+\frac{9}{x}+\frac{9}{x^2}-\frac{6}{5}\epsilon\left(-\frac{3}{x}-\frac{9}{x^2}\right)\right]. \end{aligned}$$

If $$-\lambda^2\nu = in, \quad n_1 = \sqrt{\frac{n}{2\nu}};$$

then $$x = i\lambda a = n_1 a(1+i);$$

therefore

$$Z = \tfrac{1}{2}m.in.Ve^{int}\left[1+\frac{9}{2n_1a}(1-i)-\frac{9i}{2n_1^2a^2}\right.$$
$$\left.-\frac{6}{5}\epsilon\left\{1-\frac{3}{2n_1a}(1-i)+\frac{9i}{2n_1^2a^2}\right\}\right]$$
$$=\tfrac{1}{2}m\left[\left\{1+\frac{9}{2n_1a}-\frac{6}{5}\epsilon\left(1-\frac{3}{2n_1a}\right)\right\}\frac{d}{dt}+\frac{9n}{2n_1a}\left(1+\frac{1}{n_1a}\right)\right.$$
$$\left.+\frac{9n}{5n_1a}\epsilon\left(1+\frac{3}{n_1a}\right)\right]Ve^{int},$$

since $$in.Ve^{int} = \frac{d}{dt}(Ve^{int}).$$

If we put $$1+\frac{9}{2n_1a} = L, \quad \frac{6}{5}\left(1-\frac{3}{2n_1a}\right) = L',$$
$$\frac{9n}{2n_1a}\left(1+\frac{1}{n_1a}\right) = K, \quad \frac{9n}{5n_1a}\left(1+\frac{3}{n_1a}\right) = K',$$

and l be the length of the pendulum, supposed large in comparison with a, the equation of motion of the spheroid is

$$\left[M+\tfrac{1}{2}m(L-\epsilon L')\right]\ddot{\theta}+\tfrac{1}{2}m(K+\epsilon K')\dot{\theta}+(M-m)\frac{g}{l}\theta = 0.$$

The solution of this is

$$\theta = A\epsilon^{-\delta t}\sin(nt+a),$$

where $$\delta = \frac{1}{2}\cdot\frac{K+\epsilon K'}{2M/m+L-\epsilon L'} = \frac{9n(1+n_1a)}{2n_1a\{9+(1+2M/m)2n_1a\}}$$
$$\times\left[1+\frac{2}{5}\epsilon\left\{\frac{3+n_1a}{1+n_1a}+\frac{6n_1a-9}{9+2n_1a(1+2M/m)}\right\}\right].$$

And the ratio of the apparent increase of mass to the true mass is

$$\frac{\delta M}{M} = \frac{1}{2}\cdot\frac{m}{M}(L-\epsilon L') = \frac{m}{2M}\left[1+\frac{9}{2n_1a}-\frac{6}{5}\epsilon\left(1-\frac{3}{2n_1a}\right)\right].$$

Since ν is a small quantity, and therefore n_1 a fairly large quantity, the effect of the second term of the equation of motion is small, *i.e.*, the vibrations are damped slowly; *e.g.*, taking ν, for water, in C.G.S. units = ·013, σ = density of spheroid = 8, we find

τ	a	$\frac{\delta M}{M}$	δ^{-1}
2 sec	2	$\cdot0753-\cdot0669\epsilon$	$51\cdot16-25\cdot65\epsilon$
	4	$\cdot0690-\cdot0709\epsilon$	$103\cdot98-53\cdot13\epsilon$
5 sec	2	$\cdot0825-\cdot0609\epsilon$	$80\cdot06-37\cdot35\epsilon$
	4	$\cdot0725-\cdot0725\epsilon$	$164\cdot10-81\cdot35\epsilon$

16. The corresponding results for an oblate spheroid are easily found. The solution of the equation of motion is

$$\psi = \psi_1 + \psi_2,$$

where ψ_1, ψ_2 satisfy the equations

$$\left[\frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}-\tanh\alpha\,\frac{d}{d\alpha}-\cot\beta\,\frac{d}{d\beta}\right]\psi_1 = 0,$$

$$\left[\frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}-\tanh\alpha\,\frac{d}{d\alpha}-\cot\beta\,\frac{d}{d\beta}+\lambda^2c^2(\sinh^2\alpha+\cos^2\beta)\right]\psi_2 = 0.$$

And the pressure at any point is given by

$$-dQ = \frac{1}{c\cosh\alpha\sin\beta}\,\frac{d}{dt}\left(\frac{d\psi_1}{d\alpha}d\beta-\frac{d\psi_1}{d\beta}d\alpha\right).$$

If $Ve^{-\lambda^2\nu t}$ be the velocity of the spheroid along its axis, the surface conditions, supposing the fluid to stick, are

$$\left.\begin{aligned}\frac{d\psi}{d\alpha} &= Ve^{-\lambda^2\nu t}\,.\,c^2\sinh\alpha_0\cosh\alpha_0\sin^2\beta,\\ \frac{d\psi}{d\beta} &= Ve^{-\lambda^2\nu t}\,.\,c^2\cosh^2\alpha_0\sin\beta\cos\beta.\end{aligned}\right\}$$

Writing ζ for $\sinh\alpha$, and μ for $\cos\beta$, in the equation for ψ_1 we get

$$(\zeta^2+1)\frac{d^2\psi_1}{d\zeta^2}+(1-\mu^2)\frac{d^2\psi_1}{d\mu^2} = 0,$$

the solution of which is

$$\psi_1 = \Sigma A'_n \epsilon^{-\lambda^2\nu t}(1-\mu^2)\frac{dP_n}{d\mu}(1+\zeta^2)\frac{dq_n}{d\zeta},$$

where P_n, q_n are zonal harmonics of μ and $i\zeta$ of the first and second kinds.

Writing, in the equation for ψ_2, ξ for $\lambda c \sinh \alpha$, and μ for $\cos\beta$, we see that the solution will be the same function of ξ and μ as our former one was of η and μ, except that in the equations which determine $a_0, a_1, \ldots, -\lambda^2c^3$ is written for λ^2c^2.

The force resisting the motion of the spheroid is now

$$Z = \pi\rho\lambda^2\nu\left[c\cosh\alpha_0\int_{-1}^{+1}\frac{d\psi_1}{d\alpha}\,d\mu + 2c\sinh\alpha_0\int_{-1}^{+1}\psi_2 d\mu\right]$$

$$= \tfrac{8}{3}\pi\rho\lambda^2\nu\,.\,\epsilon^{-\lambda^2\nu t}\,.\,c\left[A_0^1 + \tfrac{1}{2}Vc^2\sinh\alpha_0\cosh^2\alpha_0\right],$$

an expression of the same form as our former one.

Thursday, March 12th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. E. Parnall Culverwell, M.A., Fellow and Tutor of Trinity College, Dublin, was elected a member.

Dr. Hirst drew the attention of the members present to the loss the mathematical world had sustained by the recent death of Madame Sophie de Kowalevski,* and gave many personal reminiscences. The President also touched briefly upon some of her mathematical work.

The following communications were made:—

On Cusp-loci, which are enveloped by the Tangents at the Cusps: Prof. M. J. M. Hill.

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* Born at Moscow 15/27 December, 1853; died 10 Feb., 1891. A short account, with a list of her works, is given in the *Rendiconti del Circolo Matematico di Palermo, Tomo* v., *pp.* 121–128. See also *Nature*, Vol. XLIII., pp. 375, 376 (Feb. 19th, 1891).

On the Partitions of a Polygon: Prof. Cayley.

Some Theorems concerning Groups of Totitives of n: Prof. Lloyd Tanner.

Mr. Love and Prof. Hill spoke at some length on the Motion of a Liquid Ellipsoid under its own attraction, with special reference to Four Theorems concerning the Surfaces which always contain the same Particles.

Dr. Larmor made an impromptu communication on the Collision of Spherical Bodies.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xlix., No. 197.

"Educational Times," for March.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. iv., Nos. 1 and 2.

"Proceedings and Transactions of the Nova Scotian Institute of Natural Science," Vol. vii., Part iii.; Halifax, 1889.

"Transactions of the Canadian Institute," Vol. i., Part i.; Toronto, Oct., 1890.

"Transactions of the Connecticut Academy," Vol. viii., Part i.; Newhaven, 1890.

"Annals of Mathematics," Vol. v., No. 5.

"Bulletin des Sciences Mathématiques," Tome xv., Jan., 1891.

"Bulletin de la Société Mathématique de France," Tome xviii., Nos. 5 and 6.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stück 2.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxiv., 4 and 5 Livr.; Harlem, 1891.

"Annales de l'École Polytechnique de Delft," Tome vi., 2 Livr.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 123 and 124.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vii., Fasc. 1-4, Roma, 1891.

"Sitzungberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," xli.-liii.

"Acta Mathematica," xiv., 3; Stockholm.

"Annali di Matematica," Tomo xix., Fasc. 1; Milano.

"Theory of Differential Equations," Pt. i.; "Exact Equations and Pfaff's Problem," by Dr. A. R. Forsyth, F.R.S. Cambridge University Press, 1890. (Presented by Mr. Tucker.)

On Node- and Cusp-Loci, which are enveloped by the Tangents at the Cusps. *By* M. J. M. Hill, M.A., Sc.D., Professor of Mathematics at University College, London.

[*Read March 12th*, 1891.]

Introduction.

In a paper entitled "On the c- and p-Discriminants of Ordinary Integrable Differential Equations of the First Order," published in Vol. XIX. of the *Proceedings of the London Mathematical Society*, pp. 561–589, the conditions which hold when the curves

$$f(x, y, c) = 0$$

(where f is a rational integral indecomposable function of x, y, c) have a cusp-locus, are investigated.

It is shown further that when these conditions hold, the c-discriminant of

$$f(x, y, c) = 0,$$

and all its partial differential coefficients of the first and second order with regard to x, y, vanish when x, y are replaced by the coordinates of any point on the cusp-locus.

It follows that, if $C = 0$ be the equation of the cusp-locus, C^3 is a factor of the c-discriminant.

It is shown in the course of the investigation that the vanishing of the second differential coefficients of the c-discriminant with regard to x, y depends on the vanishing of certain fractions, all of which have a common denominator, viz., $\frac{\partial^2 f}{\partial c^2}$; but the possibility of the vanishing of the denominator is not considered.

The object of this paper is to consider those cusp-loci in which the above-mentioned common denominator vanishes; and it is shown—

(1) That C^4 is now a factor of the discriminant instead of C^3;

(2) That in this case the tangent at the cusp always touches the cusp-locus, so that the cusp-locus is also an envelope.

Geometrical considerations now point to the conclusion that if a node-locus exist such that one of the tangents at the node also touches the node-locus, then if the equation of the node-locus be $N = 0$, N^3 will be a factor of the discriminant.* This result is also demonstrated.

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* The author is indebted to one of the referees for the suggestion that this case should be investigated.

Art. 1.—*To prove that, when* $\dfrac{\partial^2 f}{\partial c^2} = 0$ *at every point of the cusp-locus, C^4 is a factor of the discriminant.*

Adopting the notation of Art. 4 of the paper above mentioned, let

$$f(x, y, c) \equiv a_m c^m + a_{m-1} c^{m-1} + \ldots + a_1 c + a_0,$$

where $a_0, a_1, \ldots a_m$ are functions of x, y, not c.

Then, if ξ, η be the coordinates of the cusp on the curve

$$f(x, y, \gamma) = 0,$$

the conditions satisfied by ξ, η, γ are

$$f(\xi, \eta, \gamma) = 0 \text{ ....(1)},$$

$$\frac{\partial f(\xi, \eta, \gamma)}{\partial \xi} = 0 \text{ ....(2)},$$

$$\frac{\partial f(\xi, \eta, \gamma)}{\partial \eta} = 0 \text{ ....(3)},$$

$$\frac{\partial f(\xi, \eta, \gamma)}{\partial \gamma} = 0 \text{ ....(4)},$$

and denoting partial differential coefficients with regard to ξ, η, γ by suffixes only, instead of by ∂ as above,

$$\left.\begin{array}{l} f_{\xi\xi} : f_{\xi\eta} : f_{\xi\gamma} \\ = f_{\xi\eta} : f_{\eta\eta} : f_{\eta\gamma} \\ = f_{\xi\gamma} : f_{\eta\gamma} : f_{\gamma\gamma} \end{array}\right\} \text{ .... (5)}.$$

To these it is necessary now to add the condition

$$f_{\gamma\gamma} = 0 \text{ ....(6)},$$

which makes the denominators of the second differential coefficients of the c-discriminant, called Ξ in the paper above mentioned, vanish.

The discriminant Ξ is formed by eliminating c between the equations

$$f(x, y, c) = 0 \text{ ....(7)},$$

and

$$\frac{\partial f(x, y, c)}{\partial c} = 0 \text{ ....(8)}.$$

The last equation gives $(m-1)$ values for c, each of which is some function of x, y. If, moreover, one of the roots, c_1, be such a function of x, y as to make at the same time

$$\frac{\partial f(x, y, c_1)}{\partial c_1} = 0 \quad\text{..............................}(9),$$

and

$$\frac{\partial^2 f(x, y, c_1)}{\partial c_1^2} = 0 \quad\text{..............................}(10),$$

then two of the roots of the equation (8) are equal. Now

$$x = \xi, \quad y = \eta, \quad c_1 = \gamma$$

satisfy the equations (9) and (10), which are the same as equations (4) and (6), when

$$x = \xi, \quad y = \eta, \quad c_1 = \gamma.$$

Hence, in forming the discriminant Ξ, it must be borne in mind that two of the roots of (8), say c_1 and c_2, are such functions of x, y that they will become equal whenever

$$x = \xi, \quad y = \eta,$$

the coordinates of a cusp on any one of the curves

$$f(x, y, c) = 0,$$

say the curve

$$f(x, y, \gamma) = 0.$$

In this case

$$c_1 = c_2 = \gamma.$$

The discriminant Ξ is now

$$a_m^{m-1} f(x, y, c_1) f(x, y, c_2) f(x, y, c_3) \dots f(x, y, c_{m-1}),$$

which will for brevity be written

$$\Xi = Q \cdot f(x, y, c_1) f(x, y, c_2),$$

or still more shortly

$$\Xi = Q \cdot f_1 \cdot f_2.$$

Now, let δ denote partial differentiation when x, y are the only independent variables, it being supposed that c has been expressed as a function of x, y;

then

$$\frac{\delta f_1}{\delta x} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial c_1}\frac{\delta c_1}{\delta x}$$

$$= \frac{\partial f_1}{\partial x},$$

since c_1 satisfies (8).

In like manner
$$\frac{\delta f_2}{\delta x}=\frac{\partial f_2}{\partial x},$$
$$\frac{\delta f_1}{\delta y}=\frac{\partial f_1}{\partial y},$$
$$\frac{\delta f_2}{\delta y}=\frac{\partial f_2}{\partial y};$$

therefore
$$\begin{aligned}\frac{\delta \Xi}{\delta x} &= \frac{\delta Q}{\delta x}\cdot f_1\cdot f_2\\ &\quad + Q\cdot\frac{\partial f_1}{\partial x}\cdot f_2 + Q\cdot\frac{\partial f_2}{\partial x}\cdot f_1,\end{aligned}$$

$$\begin{aligned}\frac{\delta^2 \Xi}{\delta x^2} &= \frac{\delta^2 Q}{\delta x^2}\cdot f_1 f_2 + 2\,\frac{\delta Q}{\delta x}\left(\frac{\partial f_1}{\partial x}f_2+\frac{\partial f_2}{\partial x}f_1\right)\\ &\quad + Q\left\{\begin{array}{l} \dfrac{\partial^2 f_1}{\partial x^2}f_2+\dfrac{\partial^2 f_1}{\partial x\,\partial c_1}\,\dfrac{\delta c_1}{\delta x}\,f_2\\ +2\dfrac{\partial f_1}{\partial x}\,\dfrac{\partial f_2}{\partial x}\\ +\dfrac{\partial^2 f_2}{\partial x^2}f_1+\dfrac{\partial^2 f_2}{\partial x\,\partial c_2}\,\dfrac{\delta c_2}{\delta x}\,f_1 \end{array}\right\},\end{aligned}$$

$$\begin{aligned}\frac{\delta^3 \Xi}{\delta x^3} &= \frac{\delta^3 Q}{\delta x^3}\cdot f_1\cdot f_2\\ &\quad + 3\,\frac{\delta^2 Q}{\delta x^2}\left(\frac{\partial f_1}{\partial x}f_2+\frac{\partial f_2}{\partial x}f_1\right)\\ &\quad + 3\,\frac{\delta Q}{\delta x}\left\{\begin{array}{l} \left(\dfrac{\partial^2 f_1}{\partial x^2}+\dfrac{\partial^2 f_1}{\partial x\,\partial c_1}\,\dfrac{\delta c_1}{\delta x}\right)f_2\\ +2\dfrac{\partial f_1}{\partial x}\,\dfrac{\partial f_2}{\partial x}\\ +\left(\dfrac{\partial^2 f_2}{\partial x^2}+\dfrac{\partial^2 f_2}{\partial x\,\partial c_2}\,\dfrac{\delta c_2}{\delta x}\right)f_1 \end{array}\right\}\\ &\quad + Q\left\{\begin{array}{l} f_2\,\dfrac{\delta}{\delta x}\left[\dfrac{\partial^2 f_1}{\partial x^2}+\dfrac{\partial^2 f_1}{\partial x\,\partial c_1}\,\dfrac{\delta c_1}{\delta x}\right]\\ +3\,\dfrac{\partial f_2}{\partial x}\left[\dfrac{\partial^2 f_1}{\partial x^2}+\dfrac{\partial^2 f_1}{\partial x\,\partial c_1}\,\dfrac{\delta c_1}{\delta x}\right]\\ +3\,\dfrac{\partial f_1}{\partial x}\left[\dfrac{\partial^2 f_2}{\partial x^2}+\dfrac{\partial^2 f_2}{\partial x\,\partial c_2}\,\dfrac{\delta c_2}{\delta x}\right]\\ +f_1\,\dfrac{\delta}{\delta x}\left[\dfrac{\partial^2 f_2}{\partial x^2}+\dfrac{\partial^2 f_2}{\partial x\,\partial c_2}\,\dfrac{\delta c_2}{\delta x}\right] \end{array}\right\}.\end{aligned}$$

Now when $x = \xi, \quad y = \eta, \quad c_1 = c_2 = \gamma;$

therefore
$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial x} = 0,$$
$$\frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial y} = 0.$$

And since also $f_1 = 0, \quad f_2 = 0,$

it follows that, in general,
$$\Xi = 0, \quad \frac{\delta \Xi}{\delta x} = 0, \quad \frac{\delta^2 \Xi}{\delta x^2} = 0, \quad \frac{\delta^3 \Xi}{\delta x^3} = 0.$$

(The conclusion will fail in the case of any of the quantities multiplying $f_1, f_2, \dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial x}$, becoming infinite. See the discussion of the case of the node-locus below.)

Similarly, if the other partial differential coefficients be worked out up to the third order, every term of each must contain one of the vanishing quantities, $f_1, f_2, \dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_1}{\partial y}, \dfrac{\partial f_2}{\partial y}$. Hence all these differential coefficients vanish.

Hence Ξ contains C^4 as a factor.

Art. 2.—*To investigate the geometrical meaning of the condition*
$$\frac{\partial^2 f}{\partial c^2} = 0.$$

By means of (5), $\qquad f_{\xi\xi} f_{\eta\eta} - (f_{\xi\eta})^2 = 0 \quad \text{..........................(11)}.$

Substituting from (6), it follows that
$$f_{\xi\gamma} = 0.$$

Similarly, $\qquad f_{\eta\gamma} = 0.$

Now, if $\xi + \delta\xi, \; \eta + \delta\eta$ be the coordinates of the cusp on the curve
$$f(x, y, \gamma + \delta\gamma) = 0,$$

then $\qquad f_{\xi\xi}\delta\xi + f_{\xi\eta}\delta\eta + f_{\xi\gamma}\delta\gamma = 0;$

this becomes $\qquad f_{\xi\xi}\delta\xi + f_{\xi\eta}\delta\eta = 0.$

Hence the tangent to the cusp-locus is
$$f_{\xi\xi}(X - \xi) + f_{\xi\eta}(Y - \eta) = 0 \quad \text{.................. ...(12)},$$

X and Y being current coordinates.

Now the tangents to the curve

$$f(x, y, \gamma) = 0$$

at ξ, η, are $\quad f_{\xi\xi}(X-\xi)^2+2f_{\xi\eta}(X-\xi)(Y-\eta)+f_{\eta\eta}(Y-\eta)^2 = 0.$

But, by means of (5), these coincide with

$$f_{\xi\xi}(X-\xi)+f_{\xi\eta}(Y-\eta) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Hence the tangent to the curve

$$f(x, y, \gamma) = 0,$$

at the cusp ξ, η on it, touches the cusp-locus at ξ, η.

Conversely, if the tangent to the cusp-locus is always the tangent at the cusp, whose equation is (13), it will follow that

$$f_{\gamma\gamma} = 0;$$

for, if $\xi+\delta\xi$, $\eta+\delta\eta$ be the coordinates of the cusp on the curve

$$f(x, y, \gamma+\delta\gamma) = 0,$$

then equations (2), (3), (4) must hold when ξ, η, γ are replaced by $\xi+\delta\xi$, $\eta+\delta\eta$, $\gamma+\delta\gamma$ respectively. Hence

$$f_{\xi\xi}\delta\xi+f_{\xi\eta}\delta\eta+f_{\xi\gamma}\delta\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

$$f_{\xi\eta}\delta\xi+f_{\eta\eta}\delta\eta+f_{\eta\gamma}\delta\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$f_{\xi\gamma}\delta\xi+f_{\eta\gamma}\delta\eta+f_{\gamma\gamma}\delta\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

But these relations, by (5), are equivalent to one only. Take (14) as this one.

Then $\qquad X = \xi+\delta\xi, \quad Y = \eta+\delta\eta,$

must satisfy the equation of the tangent (1).

Hence $\qquad f_{\xi\xi}\delta\xi+f_{\xi\eta}\delta\eta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$

Comparing with (14), $\qquad f_{\xi\gamma} = 0.$

Now (14), (15), (16) are equivalent to one equation only.

Hence $\qquad f_{\eta\gamma} = 0,$

$$f_{\gamma\gamma} = 0.$$

Hence, if the tangent to the cusp-locus is the tangent at the cusp,

$$f_{\gamma\gamma} = 0.$$

Now, such cusp-loci do not occur in general, hence the general theorem given in the paper above mentioned is not affected; but the analysis is now complete, the consequences of the vanishing of the denominator in the second differential coefficients of Ξ having been investigated.

As the cusp-locus is now an envelope, it is apparent that the factor C appears in the discriminant three times because $C = 0$ is a cusp-locus, and once because it is an envelope.

The above theory is illustrated by Examples 1, 2, and 3 below.

Art. 3.—*To prove that, if the node-locus be also an envelope, $\dfrac{\partial^2 f}{\partial c^2} = 0$, and conversely.*

At every point of the node-locus the equations (14), (15), and (16) hold.

If, now, the node-locus touch the curve, then the value of $\delta\eta/\delta\xi$, found by eliminating $\delta\gamma$ between any two of these equations, must satisfy the equation

$$f_{\xi\xi}\delta\xi^2 + 2f_{\xi\eta}\delta\xi\delta\eta + f_{\eta\eta}\delta\eta^2 = 0.$$

Multiply equation (14) by $\delta\xi$, equation (15) by $\delta\eta$, add and make use of the above; therefore

$$\delta\gamma\left[f_{\xi\gamma}\delta\xi + f_{\eta\gamma}\delta\eta\right] = 0;$$

therefore

$$f_{\xi\gamma}\delta\xi + f_{\eta\gamma}\delta\eta = 0;$$

therefore, from equation (16),

$$f_{\gamma\gamma} = 0.$$

Conversely, if

$$f_{\gamma\gamma} = 0,$$

then, from equation (16),

$$f_{\xi\gamma}\delta\xi + f_{\eta\gamma}\delta\eta = 0.$$

Hence, from equations (14) and (15),

$$f_{\xi\xi}\delta\xi^2 + 2f_{\xi\eta}\delta\xi\delta\eta + f_{\eta\eta}\delta\eta^2 = 0.$$

Hence $\delta\eta/\delta\xi$ is equal to the value of the tangent of the inclination to the axis of x of one of the tangents at the double point.

Hence the node-locus touches the curve.

Art. 4.—*To prove that, when* $\frac{\partial^2 f}{\partial c^2} = 0$ *at every point of the node-locus,* N^2 *is a factor of the discriminant.*

In this case
$$\frac{\partial f}{\partial c} = 0, \quad \frac{\partial^2 f}{\partial c^2} = 0.$$

Hence
$$\frac{\partial f}{\partial c} = 0$$
has equal roots for c.

To determine $\frac{\delta c}{\delta x}, \frac{\delta c}{\delta y}$ for each of these roots,
$$\frac{\partial^2 f}{\partial x \partial c} + \frac{\partial^2 f}{\partial c^2}\frac{\delta c}{\delta x} = 0,$$
$$\frac{\partial^2 f}{\partial y \partial c} + \frac{\partial^2 f}{\partial c^2}\frac{\delta c}{\delta y} = 0.$$

Hence, unless both
$$\frac{\partial^2 f}{\partial x \partial c} = 0, \qquad \frac{\partial^2 f}{\partial y \partial c} = 0,$$
all along the node-locus, either
$$\frac{\delta c}{\delta x} = \infty, \text{ or } \frac{\delta c}{\delta y} = \infty.$$

Now, in the case of the cusp-locus which was previously investigated,
$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial c^2} - \left(\frac{\partial^2 f}{\partial x \partial c}\right)^2 = 0,$$
when
$$x = \xi, \quad y = \eta,$$
and therefore
$$\frac{\partial^2 f}{\partial x \partial c} = 0;$$
similarly,
$$\frac{\partial^2 f}{\partial y \partial c} = 0,$$
when
$$x = \xi, \quad y = \eta.$$
But this is not the case with the node-locus; for, if these conditions hold, then the equations (14), (15), (16) reduce to
$$f_{\xi\xi}\delta\xi + f_{\xi\eta}\delta\eta = 0,$$
$$f_{\xi\eta}\delta\xi + f_{\eta\eta}\delta\eta = 0,$$
$$0 = 0.$$

Hence
$$f_{xx}f_{yy}-f_{xy}^2=0;$$
and this is the condition that there is a cusp, which is contrary to the supposition.

Now, taking the values previously found for Ξ and its differential coefficients in Art. 1, it will be found that $\frac{\delta c_1}{\delta x}$ occurs for the first time in $\frac{\delta^2\Xi}{\delta x^2}$; hence it is not possible in this case to assert, at first sight, that $\frac{\delta^2\Xi}{\delta x^2}$ vanishes.

The coefficient of $\frac{\delta c_1}{\delta x}$ is
$$f_2\frac{\partial^2 f_1}{\partial x\partial c_1}.$$

Hence
$$f_2\frac{\partial^2 f_1}{\partial x\partial c_1}\frac{\delta c_1}{\delta x}$$
$$=-f_2\frac{\left(\frac{\partial^2 f_1}{\partial x\partial c_1}\right)^2}{\frac{\partial^2 f_1}{\partial c_1^2}},$$
at a point on the node-locus $c_2=c_1$. Hence this is
$$-f_1\frac{\left(\frac{\partial^2 f_1}{\partial x\partial c_1}\right)^2}{\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)}.$$

In this case
$$f(x, y, c)=0$$
has three roots equal when x, y is a point on the node-locus. Hence f is of the form $R(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$, where γ_1, γ_2, γ_3 are functions of x, y which become equal to each other when x, y is on the node-locus.

Now
$$\frac{\partial f}{\partial c}=\frac{\partial R}{\partial c}(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$$
$$+R\left[(c-\gamma_2)(c-\gamma_3)+(c-\gamma_1)(c-\gamma_3)+(c-\gamma_1)(c-\gamma_2)\right],$$
$$\frac{\partial^2 f}{\partial c^2}=\frac{\partial^2 R}{\partial c^2}(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$$
$$+2\frac{\partial R}{\partial c}\left[(c-\gamma_2)(c-\gamma_3)+(c-\gamma_1)(c-\gamma_3)+(c-\gamma_1)(c-\gamma_2)\right]$$
$$+2R\left[(c-\gamma_1)+(c-\gamma_2)+(c-\gamma_3)\right].$$

Hence, at a point on the node-locus,

$$\frac{f}{\left(\frac{\partial^2 f}{\partial c^2}\right)} = \frac{R\,(c-\gamma)^3}{\frac{\partial^2 R}{\partial c^2}(c-\gamma)^3 + 6\frac{\partial R}{\partial c}(c-\gamma)^2 + 6R\,(c-\gamma)}$$

ERRATA.

1. The title of my paper should have been
   "On Node- and Cusp-Loci, which are also Envelopes."
2. The evaluation of the terms

$$f_2 \bigg/ \frac{\partial^2 f_1}{\partial c_1^2}, \quad f_2 \bigg/ \left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^2, \quad \text{and} \quad f_2 \bigg/ \left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^3,$$

on pages 225–227, should have been performed thus:—

c_1, c_2, are the roots of $\qquad \dfrac{\partial f(x,\, y,\, c)}{\partial c} = 0,$

which at points on the node-locus become equal to the same thing as $\gamma_1, \gamma_2, \gamma_3$.

Hence $\qquad f_2 = R(c_2-\gamma_1)(c_2-\gamma_2)(c_2-\gamma_3)$

is of the third order of small quantities, at points on the node-locus.

The most important term in $\dfrac{\partial^2 f_1}{\partial c_1^2}$ is $2R\left[(c_1-\gamma_1)+(c_1-\gamma_2)+(c_1-\gamma_3)\right]$. It is, therefore, of the first order of small quantities, at points on the node-locus. Hence, at points on the node-locus, $f_2 \Big/ \dfrac{\partial^2 f_1}{\partial c_1^2}$ and $f_2 \Big/ \left(\dfrac{\partial^2 f_1}{\partial c_1^2}\right)^2$ vanish, but $f_2 \Big/ \left(\dfrac{\partial^2 f_1}{\partial c_1^2}\right)^3$ is finite.

July 8*th*, 1891. M. J. M. HILL.

$$= -\frac{f_1 \frac{\partial^2 f_1}{\partial x \partial c_1}}{\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^3}\left[2\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^2 \frac{\partial^3 f_1}{\partial x^2 \partial c_1} - 3\frac{\partial^2 f_1}{\partial c_1^2}\,\frac{\partial^2 f_1}{\partial x \partial c_1}\,\frac{\partial^3 f_1}{\partial x \partial c_1^2} + \left(\frac{\partial^2 f_1}{\partial x \partial c_1}\right)^2 \frac{\partial^3 f_1}{\partial c_1^3}\right]$$

$$= -2\frac{f_1 \frac{\partial^2 f_1}{\partial x \partial c_1}\,\frac{\partial^3 f_1}{\partial x^2 \partial c_1}}{\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)} + 3\frac{f_1 \left(\frac{\partial^2 f_1}{\partial x \partial c_1}\right)^2 \frac{\partial^3 f_1}{\partial x \partial c_1^2}}{\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^2} - \frac{f_1 \left(\frac{\partial^2 f_1}{\partial x \partial c_1}\right)^3 \frac{\partial^3 f_1}{\partial c_1^3}}{\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^3}.$$

The first term vanishes, by the previous work. The second term contains the factor

$$\frac{f_1}{\left(\dfrac{\partial^2 f_1}{\partial c_1^2}\right)^2}=\frac{R\,(c-\gamma)^3}{\left[\dfrac{\partial^2 R}{\partial c^2}(c-\gamma)^3+6\dfrac{\partial R}{\partial c}(c-\gamma)^2+6R\,(c-\gamma)\right]^2}$$

$$=\frac{R\,(c-\gamma)}{\left[\dfrac{\partial^2 R}{\partial c^2}(c-\gamma)^2+6\dfrac{\partial R}{\partial c}(c-\gamma)+6R\right]^2},$$

which vanishes when $c=\gamma$.

The third term contains as a factor

$$\frac{f_1\dfrac{\partial^3 f_1}{\partial c_1^3}}{\left(\dfrac{\partial^2 f_1}{\partial c_1^2}\right)^3}.$$

The numerator of this expression is the product of

$$R\,(c-\gamma_1)\,(c-\gamma_2)\,(c-\gamma_3),$$

and

$$\frac{\partial^3 R}{\partial c^3}(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$$

$$+3\frac{\partial^2 R}{\partial c^2}\left[(c-\gamma_2)(c-\gamma_3)+(c-\gamma_3)(c-\gamma_1)+(c-\gamma_1)(c-\gamma_2)\right]$$

$$+6\frac{\partial R}{\partial c}\left[(c-\gamma_1)+(c-\gamma_2)+(c-\gamma_3)\right]+6R.$$

The denominator is the cube of

$$\frac{\partial^2 R}{\partial c^2}(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$$

$$+2\frac{\partial R}{\partial c}\left[(c-\gamma_2)(c-\gamma_3)+(c-\gamma_3)(c-\gamma_1)+(c-\gamma_1)(c-\gamma_2)\right]$$

$$+2R\left[(c-\gamma_1)+(c-\gamma_2)+(c-\gamma_3)\right].$$

When $\gamma_1=\gamma_2=\gamma_3$, the numerator becomes

$$R\,(c-\gamma)^3\left[\frac{\partial^3 R}{\partial c^3}(c-\gamma)^3+9\frac{\partial^2 R}{\partial c^2}(c-\gamma)^2+18\frac{\partial R}{\partial c}(c-\gamma)+6R\right],$$

and the denominator becomes the cube of

$$\frac{\partial^2 R}{\partial c^2}(c-\gamma)^3+6\frac{\partial R}{\partial c}(c-\gamma)^2+6R\,(c-\gamma).$$

Hence, dividing numerator and denominator by $(c-\gamma)^3$ and then putting $c=\gamma$, the fraction becomes $1/36R$, and this does not vanish.

The other factor of the third term, $\dfrac{\partial^2 f_1}{\partial x\partial c_1}$, has been shown above not to vanish necessarily.

Taking for f the form $R(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$,

it follows that

$$\frac{\partial^2 f}{\partial x\partial c}=\frac{\partial^2 R}{\partial x\partial c}(c-\gamma_1)(c-\gamma_2)(c-\gamma_3)$$

$$+\frac{\partial R}{\partial c}\left[-\frac{\delta\gamma_1}{\delta x}(c-\gamma_2)(c-\gamma_3)-\frac{\delta\gamma_2}{\delta x}(c-\gamma_1)(c-\gamma_3)-\frac{\delta\gamma_3}{\delta x}(c-\gamma_1)(c-\gamma_2)\right]$$

$$+\frac{\partial R}{\partial x}\left[(c-\gamma_2)(c-\gamma_3)+(c-\gamma_3)(c-\gamma_1)+(c-\gamma_1)(c-\gamma_2)\right]$$

$$+R\left[-\frac{\delta\gamma_1}{\delta x}(2c-\gamma_2-\gamma_3)-\frac{\delta\gamma_2}{\delta x}(2c-\gamma_3-\gamma_1)-\frac{\delta\gamma_3}{\delta x}(2c-\gamma_1-\gamma_2)\right].$$

Hence it would appear that, when

$$c=\gamma_1=\gamma_2=\gamma_3,$$

$\dfrac{\partial^2 f}{\partial x\partial c}$ would vanish also; but this is not necessarily the case, for $\gamma_1, \gamma_2, \gamma_3$ are irrational functions of x and y. Hence their differential coefficients may, and in some cases do, become infinite at points on the node-locus.

Hence the term $\quad f_1\left(\dfrac{\partial^2 f_1}{\partial x\partial c_1}\right)^3 \dfrac{\partial^3 f_1}{\partial c_1^3} \Big/ \left(\dfrac{\partial^2 f_1}{\partial c_1^2}\right)^3$

does not necessarily vanish on the node-locus.

In like manner the term $\dfrac{\partial f_2}{\partial x}\,\dfrac{\partial^2 f_1}{\partial x\partial c_1}\,\dfrac{\delta c_1}{\delta x}$ may not vanish on the node-locus.

Similar remarks apply to the terms containing $\dfrac{\delta c_2}{\delta x}$. (This point is specially considered in Example 4, below.) Hence $\dfrac{\delta^3 \Xi}{\delta x^3}$ does not vanish at points on the node-locus; but Ξ, $\dfrac{\delta \Xi}{\delta x}$, $\dfrac{\delta^2 \Xi}{\delta x^2}$ do vanish at points on the node-locus. Hence, if $N=0$ be the equation of the node-locus, N^3 occurs as a factor in Ξ.

EXAMPLES.

1. Consider the curves

$$y^2 = (x-c)^3.$$

To form the discriminant, it is necessary to solve the equation

$$3(x-c)^2 = 0.$$

This has two equal roots $c = x$.

Hence the discriminant $\Xi = y^4$.

Now $y = 0$ is the cusp-locus.
Moreover, the tangent at the cusp, $y = 0$, touches the cusp-locus.
Hence the occurrence of y^4 in the discriminant is accounted for.

2. Consider the curves

$$\left[\phi(x, y)\right]^2 = \left[\psi(x, y) - c\right]^3.$$

(This is a generalization of the preceding example.)

The cusp-locus is $\phi(x, y) = 0$.

The tangent to the cusp at any point ξ, η on the cusp-locus is also the tangent to the cusp-locus.

The discriminant is found by eliminating c between

$$\left[\phi(x, y)\right]^2 = \left[\psi(x, y) - c\right]^3$$

and

$$\left[\psi(x, y) - c\right]^2 = 0.$$

It is, therefore, $\Xi = \left[\phi(x, y)\right]^4$.

3. Consider the curves

$$b(c^2x - cy + a)^2 + (cy - 2a)^3 = 0,$$

where a, b are fixed constants, and c is the arbitrary parameter.

The equation is

$$c^4(bx^2) + c^3(y^3 - 2bxy) + c^2(by^2 - 6ay^2 + 2abx)$$
$$+ 2cy(6a^2 - ab) + a^2(b - 8a) = 0.$$

Now the discriminant of

$$Ac^4 + Bc^3 + Cc^2 + Dc + E = 0$$

is
$$4\,(12AE-3BD+C^2)^3$$
$$-(72ACE+9BCD-27AD^2-27EB^2-2C^3)^2.$$

Putting
$$y^2-4ax=\eta,$$
$$12AE-3BD+C^2=b\,(b-6a)\,\eta^2$$
$$72ACE+9BCD-27AD^2-27EB^2-2C^3$$
$$=b\eta^2\left[\eta\,(-2b^3+18ab)-27a^2y^2\right].$$

Hence the discriminant is
$$4b^3\,(b-6a)^3\eta^6-b^2\eta^4\left[\eta\,(-2b^3+18ab)-27a^2y^2\right]^2$$
$$=b^2\eta^4\left[4b\,(b-6a)^3\eta^2-\{\eta\,(-2b^3+18ab)-27a^2y^2\}^2\right].$$

The second factor breaks up into the two
$$\eta\,(-2b^3+18ab)-27a^2y^2\pm 2\eta b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}},$$

i.e.,
$$(y^2-4ax)\left\{-2b^3+18ab\pm 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right\}-27a^2y^2.$$

As each of these occurs only once, they correspond to envelope-loci.

As the last statement is not obvious, a direct proof will now be indicated.

If $S=0$, $T=0$ be two curves, then they will touch if it be possible to find values of x, y satisfying at the same time the equations
$$S=0,\quad T=0,$$
$$\frac{\partial S}{\partial x}\bigg/\frac{\partial T}{\partial x}=\frac{\partial S}{\partial y}\bigg/\frac{\partial T}{\partial y},$$
it being supposed that the values are not such as to make
$$\frac{\partial S}{\partial x}=0,\quad \frac{\partial S}{\partial y}=0,\quad\text{or}\quad \frac{\partial T}{\partial x}=0,\quad \frac{\partial T}{\partial y}=0,$$
which would imply that the point was a double point on the curve
$$S=0\ \text{ or }\ T=0$$
respectively.

In this case the curves are
$$b\,(c^2x-cy+a)^2+(cy-2a)^3=0\ldots\ldots\ldots\ldots\ldots\ldots\ (18),$$
$$y^2-4ax-(g+1)\,y^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ (19),$$

where
$$(g+1)\left\{-2b^3+18ab\pm 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right\}=27a^2.$$

Hence it must be possible to satisfy at the same time (18), (19) and

$$\frac{2b\,(c^2x-cy+a)\,c^2}{4a}=\frac{-2b\,(c^2x-cy+a)\,c+3c\,(cy-2a)^2}{2gy}\quad\text{......}(20).$$

Hence
$$\frac{2bc^3\,(c^2x-cy+a)}{4a}=\frac{3c^3\,(cy-2a)^2}{2gcy+4a}\quad\text{..............}\ (21);$$

therefore
$$\frac{b\,(c^2x-cy+a)}{a}=\frac{3\,(cy-2a)^2}{gcy+2a}\quad\text{..................}\ (22).$$

Squaring and using (18), this reduces to

$$9a^2(cy-2a)+b\,(gcy+2a)^2=0\quad\text{....................}(23).$$

From (19),
$$x=-gy^2/4a;$$

substituting in (22) it becomes

$$b\,(gcy+2a)\left(-\frac{c^2}{4a}gy^2-cy+a\right)=3a\,(cy-2a)^2\quad\text{.........}(24).$$

Therefore it is necessary to show that (23) and (24) are satisfied by the same value of y; but, observing that c always occurs in the form cy, it is possible to eliminate cy between the two equations.

Rearranging, the equations (23) and (24) become

$$bg^2\,(cy)^2+(4bg+\ 9a)\,a\,(cy)+2a^2\,(2b-9a)=0\ \text{......}\ (25),$$

$$bg^2\,(cy)^3+(6bg+12a)\,a\,(cy)^2$$
$$+4a^2(-bg+2b-12a)(cy)+8a^3\,(6a-b)=0\ \text{......}\ (26),$$

and by means of (25), (26) becomes

$$(2bg+3a)(cy)^2+(-4bg+4b-30a)\,a\,(cy)+8a^2\,(6a-b)=0\text{...}(27).$$

From (25) and (27), solving for the ratios $(cy)^2 : a\,(cy) : a^2$

$$\begin{vmatrix} 4bg+9a, & 2\,(2b-9a) \\ -4bg+4b-30a, & 8\,(6a-\ b) \end{vmatrix}\div(c^2y^2)$$

$$=\begin{vmatrix} 2\,(2b-9a), & bg^2 \\ 8\,(6a-b), & 2bg+3a \end{vmatrix}\div(acy)$$

$$=\begin{vmatrix} bg^2, & 4bg+9a \\ 2bg+3a, & -4bg+4b-30a \end{vmatrix}\div a^2\text{..................}\ (28).$$

Hence

$$\begin{vmatrix} 4bg+9a, & 2b-9a \\ -4bg+4b-30a, & 4(6a-b) \end{vmatrix} \begin{vmatrix} bg^2, & 4bg+9a \\ 2bg+3a, & -4bg+4b-30a \end{vmatrix}$$

$$= 2\left\{ \begin{vmatrix} 2b-9a, & bg^2 \\ 4(6a-b), & 2bg+3a \end{vmatrix} \right\}^2,$$

$$\therefore \quad \left[(60ab-8b^2)(g+1)-54a^2\right]\left[(4b^2g^2+30abg)(g+1)+27a^2\right]$$
$$+2\left[(4b^2-24ab)(g^2+g)+6ab\,(g+1)-27a^2\right]^2=0;$$

therefore
$$\left[60ab-8b^2-\frac{54a^2}{g+1}\right]\left[4b^2g^2+30abg+\frac{27a^2}{g+1}\right]$$
$$+2\left[(4b^2-24ab)\,g+6ab-\frac{27a^2}{g+1}\right]^2=0.$$

Now $\dfrac{27a^2}{g+1}=-2b^2+18ab\pm 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}$.

Hence the last equation becomes

$$\left[24ab-4b^2\mp 4b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right]\left[4b^2g^2+30abg-2b^2+18ab\pm 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right]$$
$$+2\left[(4b^2-24ab)\,g+2b^2-12ab\mp 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right]^2=0.$$

Dividing out by $4\,(b-6a)$, it becomes

$$\left\{b\pm b^{\frac{1}{2}}(b-6a)^{\frac{1}{2}}\right\}\left\{2b^2g^2+15abg-b^2+9ab\pm b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right\}$$
$$=(b-6a)\left\{2bg+b\mp b^{\frac{1}{2}}\,(b-6a)^{\frac{1}{2}}\right\}^2.$$

This reduces to $(2b^2-24ab)(g+1)^2+9ab\,(g+1)$

$$\mp b^{\frac{1}{2}}\,(b-6a)^{\frac{1}{2}}\left[2b\,(g+1)^2-9a\,(g+1)\right]=0;$$

therefore

$$2b^2-24ab+9ab\left(\frac{1}{g+1}\right)\mp b^{\frac{1}{2}}\,(b-6a)^{\frac{1}{2}}\left[2b-\frac{9a}{g+1}\right]=0;$$

therefore
$$3a\left\{2b^2-24ab\mp 2b^{\frac{3}{2}}\,(b-6a)^{\frac{1}{2}}\right\}$$
$$+\left\{b\pm b^{\frac{1}{2}}\,(b-6a)^{\frac{1}{2}}\right\}\left\{-2b^2+18ab\pm 2b^{\frac{1}{2}}\,(b-6a)^{\frac{3}{2}}\right\}=0;$$

i.e.,
$$\begin{aligned} &6ab^2-72a^2b && \mp 6ab^{\frac{3}{2}}\,(b-6a)^{\frac{1}{2}} \\ -&2b^3+18ab^2 && \pm\ 2b^{\frac{3}{2}}\,(b-6a)^{\frac{3}{2}} \\ & && \mp\ 2b^{\frac{5}{2}}\,(b-6a)^{\frac{1}{2}} \\ +&2b\,(b-6a)^2 && \pm 18ab^{\frac{3}{2}}\,(b-6a)^{\frac{1}{2}}=0; \end{aligned}$$

therefore
$$\pm 2b^{\frac{3}{2}}\,(b-6a)^{\frac{1}{2}}\left[-3a+b-6a-b+9a\right]=0.$$

As this is identically satisfied, it follows that the equations (18), (19), (20) can be satisfied by common values of x, y. The common values of y are given by equation (28). They cannot be such as to make the differential coefficients with regard to x, y of $4ax+gy^2=0$ vanish, for this curve has no singular points; and the only singular point on the curve (18) is

$$y=\frac{2a}{c},\quad x=\frac{a}{c^2}.$$

Now, the common value of y, given by equation (28), is

$$\frac{cy}{a}=\begin{vmatrix} 4bg+9a, & 2(2b-9a) \\ -4bg+4b-30a, & 8(6a-b) \end{vmatrix} \Big/ \begin{vmatrix} 2(2b-9a), & bg^2 \\ 8(6a-b), & 2bg+3a \end{vmatrix}.$$

If this give $\frac{cy}{a}=2$, then

$$\begin{vmatrix} 2b-9a, & bg^2 \\ 24a-4b, & 2bg+3a \end{vmatrix}$$

$$=\begin{vmatrix} 4bg+9a, & 2b-9a \\ -2bg+2b-15a, & 12a-2b \end{vmatrix}.$$

This reduces to $(24ab-4b^2)(g+1)^2=0$, which is not true.

Hence the common value of y is not the y of the only singular point on the curve (18).

Hence it follows that the curves (18) and (19) touch, which was to be proved.

4. Consider the curves

$$x^2+x(y-c)+(y-c)^3=0.$$

The node-locus is $x=0$;

the tangents at the node, whose coordinates are

$$x=0,\quad y=c,$$

are $x=0,\quad x+y-c=0.$

Hence the node-locus coincides with one of the tangents at the nodes. Hence the node-locus is also an envelope.

To find the discriminant Ξ, eliminate c from

$$x^2+x(y-c)+(y-c)^3=0,$$

$$x+3(y-c)^2=0.$$

The two values of c satisfying the last equation are

$$c = y \pm \sqrt{-\tfrac{1}{3}x}.$$

Hence $$\mathfrak{X} = \left[x^2 + \sqrt{-\tfrac{1}{3}x}\,(x-\tfrac{1}{3}x)\right]\left[x^2 - \sqrt{-\tfrac{1}{3}x}\,(x-\tfrac{1}{3}x)\right]$$

$$= x^4 - (-\tfrac{1}{3}x)\left(\frac{2x}{3}\right)^2$$

$$= \tfrac{4}{27}x^3 + x^4$$

$$= x^3\,(\tfrac{4}{27}+x).$$

Hence, $x = 0$, the equation of the node-locus which is also an envelope, occurs three times.

The line $$x + \tfrac{4}{27} = 0$$

cuts the curve where

$$(y-c)^3 - \tfrac{4}{27}\,(y-c) + \tfrac{16}{729} = 0.$$

Put $$y - c = \tfrac{1}{9}\eta\,;$$

therefore $$\eta^3 - 12\eta + 16 = 0\,;$$

therefore $$(\eta-2)^2\,(\eta+4) = 0.$$

Hence $$x + \tfrac{4}{27} = 0$$

cuts the curve twice where $$\eta = 2,$$

i.e., $$y - c = \tfrac{2}{9}.$$

Now the tangent at $$x = -\tfrac{4}{27},$$

$$y = c + \tfrac{2}{9},$$

is $$\left[X-(-\tfrac{4}{27})\right]\left[2\,(-\tfrac{4}{27})+\tfrac{2}{9}\right] + \left[Y-c-\tfrac{2}{9}\right]\left[-\tfrac{4}{27}+3\,(\tfrac{2}{9})^2\right] = 0\,;$$

i.e., $$(X+\tfrac{4}{27}) = 0.$$

Hence $$x + \tfrac{4}{27} = 0$$

is the ordinary envelope.

It may be observed in this case, that

$$f(x, y, c) = x^2 + x\,(y-c) + (y-c)^3,$$

$$\frac{\partial f}{\partial x} = 2x + (y-c),$$

$$\frac{\partial f}{\partial y} = x + 3\,(y-c)^2,$$

$$\frac{\partial^2 f}{\partial x^2} = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = 6\,(y-c),$$

$$\frac{\partial^2 f}{\partial x \partial c} = -1,$$

$$\frac{\partial^2 f}{\partial y \partial c} = -6\,(y-c).$$

In this case, then, at points on the node-locus, $\frac{\partial^2 f}{\partial x\, \partial c}$ does not vanish, though $\frac{\partial^2 f}{\partial y\, \partial c}$ does.

Further, the values of c, satisfying

$$\frac{\partial f}{\partial c} = 0,$$

are
$$c = y \pm \sqrt{-\tfrac{1}{3}x}\,;$$

therefore
$$\frac{\delta c}{\delta x} = \pm \tfrac{1}{2} \sqrt{-\tfrac{1}{3}}\, x^{-\frac{1}{2}},$$

$$\frac{\delta c}{\delta y} = 1.$$

Hence, at points on the node-locus, $x = 0$,

$$\frac{\delta c}{\delta x} \text{ is infinite.}$$

Now take
$$c_1 = y + \sqrt{-\tfrac{1}{3}x}, \quad c_2 = y - \sqrt{-\tfrac{1}{3}x}\,;$$

therefore
$$f(x,\, y,\, c_2)\ \frac{\partial^2 f}{\partial x \partial c_1}\ \frac{\delta c_1}{\delta x}$$

$$= \left\{x^2 + x\,(y-c_2) + (y-c_2)^3\right\}\ (-1)\left(+\tfrac{1}{2}\sqrt{-\tfrac{1}{3}}\,x^{-\frac{1}{2}}\right)$$

$$= \left(x^2 + \tfrac{2}{3}x\sqrt{-\tfrac{1}{3}x}\right)\left(-\tfrac{1}{2}\sqrt{-\tfrac{1}{3}}\,x^{-\frac{1}{2}}\right)$$

$$= -\tfrac{1}{2}\sqrt{-\tfrac{1}{3}}\,x^{\frac{3}{2}} + \tfrac{1}{6}x$$

$$= 0 \text{ at points on the node-locus.}$$

But now consider

$$f_2\frac{\delta}{\delta x}\left(\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_1}{\partial x\partial c_1}\,\frac{\delta c_1}{\delta x}\right)$$

$$=\left[x^2+x\,(y-c_2)+(y-c_2)^3\right]\frac{\delta}{\delta x}\,(2-\tfrac{1}{2}\sqrt{-\tfrac{1}{3}}\,x^{-\frac{1}{2}})$$

$$=(x^2+\tfrac{2}{3}x\sqrt{-\tfrac{1}{3}x})\,\tfrac{1}{4}\sqrt{-\tfrac{1}{3}}\,x^{-\frac{3}{2}}$$

$$=\tfrac{1}{4}\sqrt{-\tfrac{1}{3}}\,x^{\frac{1}{2}}-\tfrac{1}{18}\,;$$

and this does not vanish when $x=0$, *i.e.*, at all points of the node-locus.

Further, solving the cubic

$$x^2+x\,(y-c)+(y-c)^3=0$$

for c, it follows that

$$y-c=\sqrt[3]{-\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}+\sqrt[3]{-\frac{x^2}{2}-\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}},$$

and similar expressions.

Hence

$$\gamma_1=y-\sqrt[3]{-\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}-\sqrt[3]{-\frac{x^2}{2}-\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}$$

$$\gamma_2=y-\omega\sqrt[3]{-\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}-\omega^2\sqrt[3]{-\frac{x^2}{2}-\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}$$

$$\gamma_3=y-\omega^2\sqrt[3]{-\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}-\omega\sqrt[3]{-\frac{x^2}{2}-\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}},$$

where ω is an imaginary cube root of unity.

Now

$$\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}$$

$$=\frac{x^{\frac{3}{2}}}{3^{\frac{3}{2}}}\sqrt{1+\frac{27x}{4}}$$

$$=\frac{x^{\frac{3}{2}}}{3^{\frac{3}{2}}}\left(1+\frac{27x}{8}+\ldots\right);$$

therefore
$$\sqrt[3]{-\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}$$
$$=\left(\frac{x^{\frac{3}{2}}}{3^{\frac{3}{2}}}-\frac{x^2}{2}+\frac{3^{\frac{3}{2}}x^{\frac{5}{2}}}{8}+\dots\right)^{\frac{1}{3}}$$
$$=\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1-\frac{x^{\frac{1}{2}}3^{\frac{3}{2}}}{2}+\frac{27x}{8}+\dots\right)^{\frac{1}{3}}$$
$$=\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1-\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right).$$
$$\sqrt[3]{-\frac{x^2}{2}-\sqrt{\frac{x^4}{4}+\frac{x^3}{27}}}$$
$$=\left(-\frac{x^{\frac{3}{2}}}{3^{\frac{3}{2}}}-\frac{x^2}{2}-\frac{3^{\frac{3}{2}}x^{\frac{5}{2}}}{8}-\dots\right)^{\frac{1}{3}}$$
$$=-\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1+\frac{x^{\frac{1}{2}}3^{\frac{3}{2}}}{2}+\frac{27x}{8}+\dots\right)^{\frac{1}{3}}$$
$$=-\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1+\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right).$$

Hence
$$\gamma_1=y-\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1-\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right)$$
$$+\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1+\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right)$$
$$=y+x+\dots;$$

therefore
$$\frac{\partial\gamma_1}{\partial x}=1 \text{ when } x=0,$$
$$\gamma_2=y-\frac{\omega x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1-\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right)$$
$$+\frac{\omega^2 x^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(1+\frac{x^{\frac{1}{2}}3^{\frac{1}{2}}}{2}+\dots\right);$$

therefore
$$\gamma_2=y+\frac{x^{\frac{1}{2}}}{3^{\frac{1}{2}}}(-\sqrt{-3})-\frac{x}{2}+\dots;$$

therefore
$$\frac{\delta\gamma_2}{\delta x}=-\tfrac{1}{2}\sqrt{-1}\,.\,x^{-\frac{1}{2}}-\tfrac{1}{2}+\dots.$$

Hence $\frac{\delta\gamma_2}{\delta x}$ is infinite when $x=0$, *i.e.*, at all points on the node-locus. This is the result to which special attention was directed at the *end of Art. 4.*

On the Partitions of a Polygon. *By* Professor CAYLEY.

[*Read March* 12*th*, 1891.]

1. The partitions are made by non-intersecting diagonals; the problems which have been successively considered are (1) to find the number of partitions of an r-gon into triangles, (2) to find the number of partitions of an r-gon into k parts, and (3) to find the number of partitions of an r-gon into p-gons, r of the form $n(p-2)+2$.

Problem (1) is a particular case of (2); and it is also a particular case of (3); but the problems (2) and (3) are outside each other; for problem (3) a very elegant solution, which will be here reproduced, is given in the paper, H. M. Taylor and R. C. Rowe, "Note on a Geometrical Theorem," *Proc. Lond. Math. Soc.*, t. XIII. (1882), pp. 102–106, and this same paper gives the history of the solution of (1).

The solution of (2) is given in the memoir, Kirkman "On the k-partitions of the r-gon and r-ace," *Phil. Trans.*, t. 147 (for 1857), p. 225; viz., he there gives for the number of partitions of the r-gon into k parts (or, what is the same thing, by means of $k-1$ non-intersecting diagonals) the expression

$$\frac{[r+k-2]^{k-1}\,[r-3]^{k-1}}{[k]^{k-1}\,[k-1]^{k-1}};$$

but there is no complete demonstration of this result.

If $k=r-2$, we have the solution of the problem (1); viz., the number of partitions of the r-gon into triangles is

$$=[2r-4]^{r-3}\div[r-2]^{r-3}.$$

The present paper relates chiefly to the foregoing problem (2), the determination of the number of partitions of the r-gon into k parts, or, what is the same thing, by means of $k-1$ non-intersecting diagonals.

2. Assuming for the moment the foregoing result, then for $k=1$, the number of partitions is

$$=1,$$

for $k=2$ it is $$=\frac{r\,.\,r-3}{2},$$

for $k=3$ it is $$=\frac{r+1\,.\,r\,.\,r-3\,.\,r-4}{12},$$

and so on. As a simple verification, $k=2$, the number of partitions is equal to the number of diagonals, viz., this is number of pairs of summits *less* number of sides, that is

$$\tfrac{1}{2}r\,(r-1)-r, \quad =\tfrac{1}{2}r\,(r-3).$$

For convenience I give the Table on next page, which is a tabulation of the functions

$$U_1 = x^3+\ x^4+\ x^5+\ \ x^6+\ldots x^r,$$

$$U_2 = \qquad 2x^4+5x^5+\ 9x^6+\ldots\frac{r.r-3}{2.1}x^r,$$

$$U_3 = \qquad\qquad 5x^5+21x^6+\ldots\frac{r+1.r.r-3.r-4}{3.2.2.1}x^r,$$

$$U_4 = \qquad\qquad\qquad 14x^6+\ldots\frac{r+2.r+1.r.r-3.r-4.r-5}{4.3.2.3.2.1}x^r,$$

$$U_5 = \qquad\qquad\qquad\qquad \frac{r+3.r+2.r+1.r.r-3.r-4.r-5.r-6}{5.4.3.2.4.3.2.1}x^r.$$

&c.

3. And in connexion herewith I give the Table on page 240, which is a tabulation of the functions

$$V_2 = x^6+2x^7+\ 3x^8+\ 4x^9\ldots+1\,\frac{r-3}{1}x^{r+2},$$

$$V_3 = \qquad 4x^7+14x^8+32x^9\ldots+2\,\frac{r+1.r-3.r-4}{3.2.1}x^{r+2},$$

$$V_4 = \qquad\qquad 14x^8+72x^9\ldots+3\,\frac{r+2.r+1.r-3.r-4.r-5}{4.3.3.2.1}x^{r+2},$$

$$V_5 = \qquad\qquad\qquad 48x^9\ldots+4\,\frac{r+3.r+2.r+1.r-3.r-4.r-5.r-6}{5.4.3.4.3.2.1}x^{r+2},$$

&c.

$r=$	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13
3	1												
4	1	2											
5	1	5	5										
6	1	9	21	14									
7	1	14	56	84	42								
8	1	20	120	300	330	132							
9	1	27	225	825	1485	1287	429						
10	1	35	385	1925	5005	7007	5005	1430					
11	1	44	616	4004	14014	28028	32032	19448	4862				
12	1	54	936	7644	34398	91728	148512	143208	75582	16796			
13	1	65	1365	13650	76440	259896	556920	755820	629850	293930	58786		
14	1	77	1925	23100	157080	659736	1790712	3197700	3730650	2735810	1144066	208012	
15	1	90	2640	37400	302940	1534896	5116320	11511720	17587350	17978180	11767536	4457400	742900

$r =$	$k = 2$	3	4	5	6	7	8	9	10	11	12	13
4	1											
5	2	4										
6	3	14	14									
7	4	32	72	48								
8	5	60	225	330	165							
9	6	100	550	1320	1430	572						
10	7	154	1155	4004	7007	6006	2002					
11	8	224	2184	10192	25480	34944	24052	7072				
12	9	312	3822	22932	76440	148512	167076	100776	25194			
13	10	420	6300	47040	199920	514080	813960	775200	396800	90440		
14	11	550	9900	89760	476240	1534996	3197700	4263600	3517470	1634380	326876	
15	12	704	14960	161568	1023264	4093056	10744272	18759840	18573816	15690048	6547520	1188640

4. These functions, U and V, are particular values satisfying the equation

$$(V_2+V_3y+V_4y^2+\dots) = (U_1+U_2y+U_3y^2+U_4y^3+\dots)^2;$$

that this is so will appear from the following general investigation.

5. Taking x, y as independent variables, denoting by X an arbitrary function of x, and using accents to denote differentiations in regard to x, we require the following identity:

$$\frac{2}{1.2}X^2+\frac{4y}{1.2.3}(X^3)'+\frac{6y^2}{1.2.3.4}(X^4)''+\dots$$

$$=\left\{X+\frac{y}{1.2}(X^2)'+\frac{y^2}{1.2.3}(X^3)''+\dots\right\}^2,$$

which I prove as follows. Writing U to denote the same function of u which X is of x, I start from the equation

$$u = x+yU,$$

which determines u as a function of the independent variables x, y. We have

$$\frac{du}{dy}(1-yU') = U,\quad \frac{du}{dx}(1-yU') = 1,$$

where the accent denotes differentiation in regard to u; hence

$$\frac{du}{dy} = U\frac{du}{dx} = \frac{u-x}{y}\frac{du}{dx},\text{ or say } y\frac{du}{dy} = (u-x)\frac{du}{dx}.$$

Writing $u_1 = \int u\,dx,$

and therefore $\dfrac{du_1}{dx} = u,$

this equation may be written

$$y\frac{d^2u_1}{dx\,dy}-\frac{du_1}{dx} = u\frac{du}{dx}-u-x\frac{du}{dx};$$

or, integrating with respect to x, we have

$$y\frac{du_1}{dy}-u_1 = \tfrac{1}{2}u^2-ux,$$

or say
$$\frac{2}{y}\frac{du_1}{dy} - \frac{2(u_1-\frac{1}{2}x^2)}{y^2} = \frac{(u-x)^2}{y^2},$$

that is,
$$2\frac{d}{dy}\left(\frac{u_1-\frac{1}{2}x^2}{y}\right) = \frac{(u-x)^2}{y^2}.$$

6. But, from the equation
$$u = x+yU,$$

we have
$$u = x+yX+\frac{y^2}{1.2}(X^2)'+\frac{y^3}{1.2.3}(X^3)''+\ldots,$$

and thence
$$u_1 = \tfrac{1}{2}x^2+yX_1+\frac{y^2}{1.2}X^2+\frac{y^3}{1.2.3}(X^3)'+\ldots,$$

if for a moment X_1 is written for $\int X dx$. And hence, from the relation obtained above, we have the required identity

$$\frac{2}{1.2}X^2+\frac{4y}{1.2.3}(X^3)'+\frac{6y^2}{1.2.3.4}(X^4)''+\ldots$$
$$=\left\{X+\frac{y}{1.2}(X^2)'+\frac{y^2}{1.2.3}(X^3)''+\ldots\right\}^2.$$

This of course gives the series of identities

$$\frac{2}{1.2}X^2 = X^2,$$
$$\frac{4}{1.2.3}(X^3)' = \frac{2}{1.2}X(X^2),$$
$$\frac{6}{1.2.3.4}(X^4)'' = \frac{2}{1.2.3}X(X^3)''+\left\{\frac{1}{1.2}(X^2)'\right\}^2;$$
$$\vdots$$

or say
$$X^2 = X^2,$$
$$(X^3)' = \tfrac{3}{2}X(X^2)',$$
$$(X^4)'' = \tfrac{4}{3}X(X^3)''+\{(X^2)'\}^2,$$
$$\vdots$$

any of which may be easily verified.

7. I multiply each side of the identity by x^2, and write

$$U_1 = x \, . \, X, \qquad\qquad V_2 = x^2 \frac{2}{1.2} X^2,$$

$$U_2 = x \frac{1}{1.2} (X^2)', \qquad\qquad V_3 = x^2 \frac{4}{1.2.3} (X^3)',$$

$$U_3 = x \frac{1}{1.2.3} (X^3)'', \qquad\qquad V_4 = x^2 \frac{6}{1.2.3.4} (X^4)'',$$

$$U_4 = x \frac{1}{1.2.3.4} (X^4)''', \qquad\qquad V_5 = x^2 \frac{8}{1.2.3.4.5} (X^5)'''.$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

We thus obtain two sets of functions U, V, satisfying the before-mentioned equation. We have

$$(V_2 + yV_3 + y^2 V_4 + \ldots) = (U_1 + yU_2 + y^2 U_3 + \ldots)^2;$$

and it will be observed that we have, moreover, the relations

$$U_2 = \tfrac{1}{2} x \, (x^{-2} V_2)', \quad U_3 = \tfrac{1}{4} x \, (x^{-2} V_3)', \quad U_4 = \tfrac{1}{6} x \, (x^{-2} V_4), \ldots$$

8. In particular, if $\qquad X = \dfrac{x^2}{1-x},$

then the general term

in X is x^{r-1}, the first term occurring when $r = 3, \ldots$

in X^2 is $(r-3)\, x^r$, „ „ $r = 4, \ldots$

in X^3 is $\dfrac{r-3 \, . \, r-4}{1.2} x^{r+1}$, „ „ $r = 5, \ldots$

in X^4 is $\dfrac{r-3 \, . \, r-4 \, . \, r-5}{1.2.3} x^{r+2}$, „ „ $r = 6, \ldots$

$\vdots$

from which it appears that, for this value of X, U_1, U_2, U_3, U_4, &c. have the before-mentioned values (No. 2), and further that V_2, V_3, V_4, V_5, &c. have also the before-mentioned values (No. 3).

9. We do not absolutely require, but it is interesting to obtain, the finite expressions of these functions. We have

$$
\begin{aligned}
(1-x)\ U_1 &= x^3\,(1),\\
(1-x)^3\ U_2 &= x^4\,(2-x),\\
(1-x)^5\ U_3 &= x^5\,(5-4x+x^2),\\
(1-x)^7\ U_4 &= x^6\,(14-14x+6x^2-x^3),\\
(1-x)^9\ U_5 &= x^7\,(42-48x+27x^2-8x^3+x^4),\\
(1-x)^{11} U_6 &= x^8\,(132-165x+110x^2-44x^3+10x^4-x^5)\,;\\
&\vdots\\
(1-x)^2\ V_2 &= x^6\,(1),\\
(1-x)^4\ V_3 &= x^7\,(4-2x),\\
(1-x)^6\ V_4 &= x^8\,(14-12x+3x^2),\\
(1-x)^8\ V_5 &= x^9\,(48-54x+24x^2-4x^3),\\
(1-x)^{10} V_6 &= x^{10}\,(165-220x+132x^2-40x^3+5x^4),\\
&\vdots
\end{aligned}
$$

and here the factors in ( ) satisfy the series of relations

$$
\begin{aligned}
1 &= 1^2,\\
4-2x &= 2\,(2-x),\\
14-12x+3x^2 &= 2.1\,(5-4x+x^2)+(2-x)^2,\\
48-54x+24x^2-4x^3 &= 2.1\,(14-14x+6x^2-x^3)+2\,(2-x)(5-4x+x^2),\\
&\vdots
\end{aligned}
$$

corresponding to $V_2 = U_1^2,\quad V_3 = 2U_1U_2,$ &c.,

given by the before-mentioned equation (No. 7), between the functions V and U.

10. It is to be shown that, taking $U_1, U_2, U_3, \ldots$ for the functions which belong to the partitions of the r-gon (assumed to be unknown *functions* of r and the suffixes), and connecting them with a set of

functions $V_2, V_3, V_4, \ldots$ by the relations

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)', \text{ \&c.,}$$

then we have the foregoing identical equation

$$(V_2 + V_3y + V_4y^2 + \ldots) = (U_1 + U_2y + U_3y^2 + U_4y^3 + \ldots)^2.$$

This implies the relations

$$\begin{aligned} V_2 &= U_1^2, \\ V_3 &= 2U_1U_2, \\ V_4 &= 2U_1U_3 + U_2^2, \\ V_5 &= 2U_1U_4 + 2U_2U_3, \\ &\text{\&c.} \end{aligned}$$

Thus, if U_1 is known, the equation

$$V_2 = U_1^2$$

determines V_2, and then

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)'$$

determines U_2, so that U_1, U_2 are known; and we thence in the same way find successively U_3 and V_3, U_4 and V_4, and so on; that is, assuming only that U_1 has the before-mentioned value,

$$U_1 = x^3 + x^4 + x^5 + \ldots + x^r \ldots,$$

it follows that all the remaining functions U and V must have their before-mentioned values. But the function

$$U_1 = x^3 + x^4 + x^5 + \ldots,$$

where each coefficient is $= 1$, is evidently the proper expression for the generating function of the number of partitions of the r-gon into a single part; and we thus arrive at the proof that the remaining functions U, which are the generating functions for the number of partitions of the r-gon into 2, 3, 4, ... k, parts, have their before-mentioned values.

11. Considering, then, the partition problem from the point of view just referred to, I write A_r, B_r, C_r, ... for the number of partitions of

an r-gon into 1 part, 2 parts, 3 parts, &c., and form therewith the generating functions

$$\begin{aligned} U_1 &= A_3x^3 + A_4x^4 \quad \ldots + \quad A_rx^r \ldots, \\ U_2 &= \qquad\quad\; B_4x^4 \quad \ldots + \quad B_rx^r \ldots, \\ U_3 &= \qquad\qquad\quad C_5x^5 \ldots + \quad C_rx^r \ldots, \\ &\vdots \end{aligned}$$

and also the functions

$$\begin{aligned} V_2 &= \frac{2}{4}B_4x^6 \quad \ldots + \frac{2}{r}B_rx^{r+2} \ldots, \\ V_3 &= \qquad\quad \frac{4}{5}C_5x^7 \ldots + \frac{4}{r}C_rx^{r+2} \ldots, \\ &\vdots \end{aligned}$$

where observe that the functions U, V are such that

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)', \text{ \&c.}$$

To fix the ideas, consider an r-gon which is to be divided into six parts. Choosing any particular summit, and from this summit drawing a diagonal successively to each of the non-adjacent $r-3$ summits, we divide the r-gon into two parts in $r-3$ different ways; viz., the two parts are

a 3-gon and $(r-1)$-gon,
4-gon and $(r-2)$-gon,
⋮
$(r-1)$-gon and 3-gon;

say any one of these ways is

an α-gon and β-gon, $\alpha+\beta = r-2$.

Next, writing $a+b = 6$,

that is, $a, b = 1, 5,$
2, 4,
3, 3,
4, 2,
5, 1,

we divide in every possible way the a-gon into a parts, and the β-gon into b parts (so dividing the r-gon into six parts). Observing that A, B, C, D, E, F are the letters belonging to the numbers 1, 2, 3, 4, 5, 6, respectively, the number of parts which we thus obtain (corresponding to the different values of a, b) are

$$A_a E_\beta + B_a D_\beta + C_a C_\beta + D_a B_\beta + E_a A_\beta,$$

and summing for the different values of a, β $(a+\beta = r+2)$, the whole number of parts is

$$= \text{coeff. } x^{r+2} \text{ in } (U_1U_5 + U_2U_4 + U_3U_3 + U_4U_2 + U_5U_1),$$

that is, $\quad$ in $(2U_1U_5 + 2U_2U_4 + U_3^2)$.

12. To obtain the whole number of the partitions of the r-gon into six parts, we must perform the foregoing process successively with each summit of the r-gon as the summit from which is drawn the diagonal which divides the r-gon into two parts; that is, the number found as above is to be multiplied by r. We thus obtain all the partitions repeated a certain number of times, viz., each partition into six parts is a partition by means of five diagonals, and is thus obtainable by the foregoing process, taking any one of the ten extremities of these diagonals as the point from which is drawn the diagonal which divides the r-gon into two parts; that is, we have to divide the foregoing product by 10. The final result thus is

$$\frac{10}{r} F_r = \text{coeff. } x^{r+2} \text{ in } (2U_1U_5 + 2U_2U_4 + U_3^2),$$

where $\quad \frac{10}{r} F_r$ is $=$ coeff. x^{r+2} in V_6;

we thus have $\quad V_6 = 2U_1U_5 + 2U_2U_4 + U_3^2.$

13. The reasoning is perfectly general; and applying it successively to the partitions into two parts, three parts, &c., we have

$$V_2 = U_1^2,$$
$$V_3 = 2U_1U_2,$$
$$V_4 = 2U_1U_3 + U_2^2,$$
$$V_5 = 2U_1U_4 + 2U_2U_3,$$
$$\vdots$$

where any function V is related to the corresponding function U as above. The value of U_1 is obviously

$$U_1 = x^3+x^4+x^5+\ldots = \frac{x^3}{1-x};$$

and hence the several functions U and V have the values above written down; the general term of U_k is

$$\frac{[r+k-2]^{k-1}\,[r-3]^{k-1}}{[k]^{k-1}\,[k-1]^{k-1}}\,x^r;$$

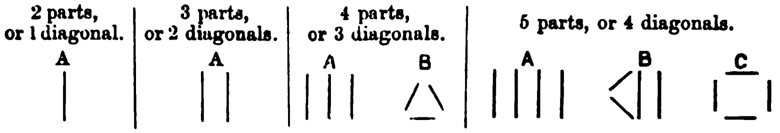
and the number of partitions of the r-gon into k parts is equal to the coefficient of x^r in this general term.

14. In the investigations which next follow, I consider, without using the method of generating functions, the problem of the partition of the r-gon into 2, 3, 4 or 5 parts; it will be convenient to state the results as follows:

Number of Partitions.

2 parts	$\frac{r}{2}\;A,$
3 parts	$\frac{r}{4}\;2A,$
4 parts	$\frac{r}{6}\,(3A+2B),$
5 parts	$\frac{r}{8}\,(4A+8B+2C);$

where the capital letters refer to different "diagonal-types," thus:



viz.: if, in a polygon divided into k parts by means of $k-1$ diagonals, we delete all the sides of the polygon, leaving only the diagonals, then these will present themselves under distinct forms, which are what I call "diagonal-types"; for instance,

$$k=4,$$

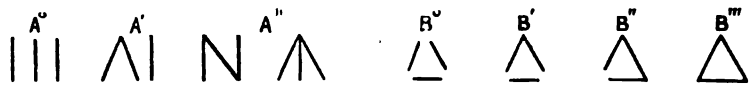
there are the two types A and B shown in the above diagram for four parts.

15. It is to be observed that we have sub-types corresponding to the coalescence of the terminal points of different diagonals; thus,

$$k = 4.$$

Writing now A° and B° to denote the forms without coalescences, we have the sub-types A°, A', A'' and B°, B, B', B'', B''', as follows:

4 parts, or 3 diagonals.



where observe that under A'' are included two distinct forms, which, nevertheless, by reason that there is in each of them the same number $(=2)$ of coalescences, are reckoned as belonging to the same sub-type.

16. The numbers called A, B, C, &c. have values which may be directly determined. I write down as follows:

1 diagonal $A = \dfrac{r-3}{1}$,

2 diagonals $A = \dfrac{r-3.r-2.r-1}{6} - \dfrac{r-3}{1} = \dfrac{r-3.r-4}{6}(r+1)$.

$$\begin{aligned} \text{Calculation is}\quad r-2.r-1 &= r^2-3r+2 \\ -6 \qquad & \quad\;\; -6 \\ \hline & r^2-3r-4 \\ &= r-4.r+1. \end{aligned}$$

3 diagonals

$$A = \frac{r-3.r-2.r-1.r.r+1}{120} - 2\left(\frac{r-3.r-2.r-1}{6} - \frac{r-3}{1}\right) - \frac{r-3}{1}$$

$$= \frac{r-3.r-2.r-1.r.r+1}{120} - 2\frac{r-3.r-2.r-1}{6} + 1\frac{r-3}{1}$$

$$= \frac{r-3.r-4.r-5}{120}(r^2+7r+2).$$

Calculation is

$$\begin{array}{lr}
r-2.r-1.r.r+1 = & r^4-2r^3-r^2+2r \\
\quad -40.r-2.r-1 & -40r^2+120r-80 \\
\qquad\qquad +120 & +120 \\
\hline
 & r^4-2r^3-41r^2+122r+40 \\
 & = r-4.r-5.r^2+7r+2.
\end{array}$$

$$B=\frac{r-5.r-4.r-3.r-2.r-1}{120}$$

$$=\frac{r-3.r-4.r-5}{120}\,(r-1.r-2)$$

4 diagonals $A=\dfrac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}$

$$-3\frac{r-3.r-2.r-1.r.r+1}{120}+3\frac{r-3.r-2.r-1}{6}-1\frac{r-3}{1}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}\,(r^3+18r^2+65r).$$

Calculation is

$$\begin{array}{lr}
r-2.r-1.r.r+1.r+2.r+3= & r^6+3r^5-5r^4-15r^3+4r^2+12r \\
\quad -126.r-2.r-1.r.r+1 & -126r^4+252r^3+126r^2-252r \\
\qquad +2520.r-2.r-1 & +2520r^2-7560r+5040 \\
\qquad\qquad -5040 & -5040 \\
\hline
 & r^6+3r^5-131r^4+237r^3+2650r^2-7800r \\
 & = r-4.r-5.r-6.r^3+18r^2+65r.
\end{array}$$

$$B=\frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040}-\frac{r-5.r-4.r-3.r-2.r-1}{120}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}\,(r-1.r-2.r+7),$$

$$C=\frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}\,(r-1.r-2.r-7).$$

Calculation is

$$\begin{array}{lr}
r.r+1 & =r^2+r \\
-42 & -42 \\
\hline
 & r^2+r-42 \\
 & =r-6.r+7.
\end{array}$$

17. To explain the formation of these expressions, observe that:

One diagonal.—There must be on each side of the diagonal, or say in each of the two "intervals" formed by the diagonal, two sides; there remain $r-4$ sides which may be distributed at pleasure between the two intervals, and the number of ways in which this can be done is

$$=\frac{r-3}{1}.$$

Two diagonals.—There must be on each side of the two diagonals, or say in two of the four intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same four intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1}{6}.$$

But we must exclude the distributions where there is 0 side in the one interval and 0 side in the other interval between the two diagonals; the number of these is that for the case of the coalescence of the two diagonals into a single diagonal, viz., it is

$$=\frac{r-3}{1};$$

and thus the number required is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1}.$$

18. Three diagonals, A.—There must be on each side of the three diagonals, that is, in two of the six intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same six intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1.r.r+1}{120}.$$

But we must exclude distributions which would permit the coalescence of the first and second, or of the second and third, or of all three of the diagonals. For the coalescence of the first and second diagonals (the third diagonal not coalescing) the term to be subtracted is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1};$$

and the same number for the coalescence of the second and third diagonals (the first diagonal not coalescing); that is, the last-mentioned number is to be multiplied by 2; and for the coalescence of all three diagonals the number to be subtracted is

$$=\frac{r-3}{1};$$

we have thus the foregoing value

$$\frac{r-3.r-2.r-1.r.r+1}{120}-2.\frac{r-3.r-2.r-1}{6}+1\frac{r-3}{1},$$

where it will be observed that we have the binomial coefficients 1, 2, 1 with the signs +, −, +.

Three diagonals, *B*.—There must be outside each of the three diagonals, that is, in three of the six intervals formed by the diagonals, two sides; and there remain $r-6$ sides to be distributed between the six intervals; the number of ways in which this can be done is

$$=\frac{r-5.r-4.r-3.r-2.r-1}{120};$$

and there is here no coalescence of diagonals, so that this is the number required.

19. Four diagonals, *A*.—There must be on each side of the four diagonals, that is, in two of the eight intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}.$$

But this number requires to be corrected for coalescences, as in the case Three diagonals, *A*, and the required number is thus found to be

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}-3\frac{r-3.r-2.r-1.r.r+1}{120}$$
$$+3\frac{r-3.r-2.r-1}{6}+1\frac{r-3}{1}.$$

Four diagonals, *B*.—There must be outside of three of the diagonals, that is, in each of three of the eight intervals formed by the diagonals, two sides; there remain $r-6$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040}.$$

There is a correction for the coalescence of two of the diagonals, giving rise to a form such as Three diagonals, *B*, and consequently a term

$$-\frac{r-5.r-4.r-3.r-2.r-1}{120},$$

which, with the first-mentioned term, gives the required number.

Four diagonals, *C*.—There must be outside of each of the diagonals, that is, in each of four of the eight intervals formed by the diagonals, two sides; there remain $r-8$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040},$$

which is the required number.

20. In the expressions of No. 14, A, $2A$, $3A+2B$, $4A+8B+2C$, if we regard the terminals of the diagonals as given points, then (1) we have two summits, which can be joined in one way only, giving rise to the diagonal-type A; (2) we have four summits, which can be joined in two ways only, so as to give rise to the diagonal-type A; (3) we have six summits, which can be joined in three ways so as to give rise to a diagonal-type A, and in two ways so as to give rise to a diagonal-type B; and (4) we have eight summits, which can be joined in four ways so as to give rise to a diagonal-type A, in eight ways so as to give rise to a diagonal-type B, and in two ways so as to give rise to a diagonal-type C; we have thus the linear forms in question. To obtain the number of partitions, we have in each case to multiply by r. To explain this, after the polygon is drawn, imagine the summits to be numbered 1, 2, 3, ... r in succession (the numbering may begin at any one of the r summits), and, regarding each of these numberings as giving a different partition, we should have the factor r. But in fact the partitions so obtained are not all of them distinct, but we have in each case a system of partitions repeated as many times as there are summits of the diagonals, that is, a number of times equal to twice the number of the diagonals; and we have thus, after the multiplication by r, to divide by the numbers 2, 4, 6, 8, in the four cases respectively.

21. We hence have immediately

Two parts, number of partitions

$$=\frac{r}{2}A=\frac{r.r-3}{2.1};$$

Three parts, number of partitions

$$=\frac{r}{2}A=\frac{r+1\,.\,r\,.\,r-3\,.\,r-4}{3\,.\,2\,.\,2\,.\,1}\,;$$

Four parts, number of partitions

$$=\frac{r}{6}(3A+2B)=\frac{r+2\,.\,r+1\,.\,r\,.\,r-3\,.\,r-4\,.\,r-5}{4\,.\,3\,.\,2\,.\,3\,.\,2\,.\,1},$$

the calculation being

$$\begin{array}{rl}
3\,(r^2+7r+2)= & 3r^2+21r+\ 6\\
+2\,.\,r-1\,.\,r-2 & +2r^2-\ 6r+\ 4\\
\hline
 & 5r^2+15r+10\\
= & 5\,.\,r+1\,.\,r+2\,;
\end{array}$$

Five parts, number of partitions

$$=\frac{r}{8}(4A+8B+2C)=\frac{r+3\,.\,r+2\,.\,r+1\,.\,r\,.\,r-3\,.\,r-4\,.\,r-5\,.\,r-6}{5\,.\,4\,.\,3\,.\,2\,.\,4\,.\,3\,.\,2\,.\,1},$$

the calculation being

$$\begin{array}{rl}
4\,(r^3+18r^2+65r)= & 4r^3+72r^2+260r\\
+8\,.\,r-1\,.\,r-2\,.\,r+7 & +8r^3+32r^2-152r+112\\
+2\,.\,r-1\,.\,r-2\,.\,r-7 & +2r^3-20r^2+\ 46r-\ 28\\
\hline
= & 14r^3+84r^2+154r+\ 84\\
= & 14(r^3+6r^2+\ 11r+6)\\
= & 14\,.\,r+1\,.\,r+2\,.\,r+3.
\end{array}$$

To complete the theory, it would be in the first instance necessary to find for any given number of diagonals, $k-1$, whatever, the number and form of the diagonal-types, A, B, C, &c.; this is itself an interesting question in the Theory of Partitions, but I have not considered it.

22. Although the foregoing process (which, it will be observed, deals with the diagonal-types, without any consideration of the sub-types) is the most simple for the determination of the numbers A, B, C, &c., yet it is interesting to give a second process. Considering the several cases in order:

One diagonal, A.—The diagonal has two summits; we must have on each side of it one summit, and there remain $r-4$ summits which may be distributed between the two intervals formed by the diagonals. This can be done in $\frac{r-3}{1}$ ways, or we have, as before,

$$A = \frac{r-3}{1}.$$

Two diagonals, A.—The diagonals have four summits; we must have outside each diagonal one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; this can be done in $\frac{r-5.r-4.r-3}{6}$ ways, or we have this value for A°. But the two top summits of the diagonals, or the two bottom summits, may coalesce; in either case the diagonals have three summits. We must have outside each diagonal one summit, and there remain $r-5$ summits to be distributed between the three intervals formed by the diagonals; the number of ways in which this can be done is

$$= \frac{r-4.r-3}{2},$$

say this is the value of A'. And we then have $A = A^\circ + 2A'$,

$$= \frac{r+1.r-3.r-4}{6}, \text{ as before.}$$

The calculation is $r-5+6 = r+1$.

23. Three diagonals, A.—See No. 15 for the figures of the sub-types. We have

$$A = A^\circ + 4A' + 4A'',$$

where the coefficients, 4 and 4, are the number of ways in which A' and A'' respectively can be derived from A° by coalescences of summits. For A° the diagonals have six summits, and there must be outside of two diagonals one summit; there remain $r-8$ summits to be distributed between the six intervals formed by the diagonals, and we have

$$A^\circ = \frac{r-7.r-6.r-5.r-4.r-3}{120}.$$

For A' the diagonals have five summits, and we must have outside of each of two diagonals, one summit; there remain $r-7$ summits to be distributed between the five intervals formed by the diagonals; we thus have

$$A' = \frac{r-6.r-5.r-4.r-3}{24}.$$

For A'' the diagonals have four summits; there must be outside of each of two diagonals one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; we thus have

$$A'' = \frac{r-5.r-4.r-3}{6}.$$

The foregoing values give

$$A = \frac{r-3.r-4.r-5}{120}\,(r^2+7r+2),$$

as before. The calculation is

$$\begin{array}{rr} r-6.r-7 = & r^2-13r+\ 42 \\ +20.r-6 & +20r-120 \\ +80 & +\ 80 \\ \hline & r^2+\ 7r+\ \ 2 \end{array}$$

Three diagonals, B.—See No. 15 for the figures of the sub-types. We have

$$B = B^\circ+3B'+3B''+B'''.$$

For B° the diagonals have six summits, and there must be outside each of the three diagonals one summit; there remain $r-9$ summits to be distributed between the six intervals formed by the diagonals. We thus have

$$B^\circ = \frac{r-8.r-7.r-6.r-5.r-4}{120}.$$

Similarly,

$$B' = \frac{r-7.r-6.r-5.r-4}{24},\quad B'' = \frac{r-6.r-5.r-4}{6},\quad B''' = \frac{r-5.r-4}{2}.$$

Hence $$B = \frac{r-5.r-4.r-3.r-2.r-1}{120},$$

as before. The calculation is

$$\begin{array}{rr} r-6.r-7.r-8 = & r^3-21r^2+146r-336 \\ +15.r-6.r-7 & +15r^2-195r+630 \\ +60.r-6 & +\ 60r-360 \\ +60 & +\ 60 \\ \hline & r^3-\ 6r^2+\ 11r-\ \ 6 \\ & = r-1.r-2.r-3. \end{array}$$

24. **Four diagonals, A.**—The figures of the sub-types of A, B, C can be supplied without difficulty. We have

$$A = A^\circ + 6A' + 12A'' + 8A''',$$

where I remark that the numerical coefficients 1, 6, 12, 8 are the terms of $(1, 2)^3$. We have

$$A^\circ = \frac{r-9.r-8.r-7.r-6.r-5.r-4.r-3}{5040},$$

$$A' = \frac{r-8.r-7.r-6.r-5.r-4.r-3}{720},$$

$$A'' = \frac{r-7.r-6.r-5.r-4.r-3}{120},$$

$$A''' = \frac{r-6.r-5.r-4.r-3}{24},$$

and thence $A = \dfrac{r-3.r-4.r-5.r-6}{5040}\,(r^3+18r^2+65r),$

as before. The calculation is

$$\begin{array}{rrrrr}
r-9.r-8.r-7 = & r^3 & -24r^2 & +191r & -504 \\
+42.r-8.r-7 & & +42r^2 & -630r & +2352 \\
+504.r-7 & & & +504r & -3528 \\
+1680 & & & & +1680 \\
\hline
& r^3 & +18r^2 & +65r &
\end{array}$$

Four diagonals, B.—We have

$$B = B^\circ + 5B' + 9B'' + 7B''' + 2B^{\text{iv}},$$

where the coefficients, 1, 5, 9, 7, 2, are the terms of $(1, 1)^3\,(1, 2)$. We have

$$B^\circ = \frac{r-10.r-9.r-8.r-7.r-6.r-5.r-4}{5040},$$

$$B' = \frac{r-9.r-8.r-7.r-6.r-5.r-4}{720},$$

$$B'' = \frac{r-8.r-7.r-6.r-5.r-4}{120},$$

$$B''' = \frac{r-7.r-6.r-5.r-4}{24},$$

$$B^{\text{iv}} = \frac{r-6.r-5.r-4}{6},$$

and thence $B = \frac{r-3.r-4.r-5.r-6}{5040} r-1.r-2.r+7$

as before. The calculation is

$$\begin{array}{rr} r-10.r-9.r-8.r-7 = & r^4-34r^3+431r^2-2414r+\ 5040 \\ +35\,.\,r-9.r-8.r-7 & +35r^3-840r^2+6685r-17640 \\ +378.r-8.r-7 & +378r^2-5670r+21168 \\ +1470.r-7 & +1470r-10290 \\ +1680 & +\ 1680 \\ \hline & r^4+\ r^3\ -\ 31r^2+\ \ 71r-\ \ \ \ 42 \\ & = r-3.r-2.r-1.r+7. \end{array}$$

Four diagonals, C.—We have

$$C = C^0+4C'+6C''+4C'''+1C^{\text{iv}},$$

where the coefficients are the terms of $(1, 1)^4$. We have

$$C^0 = \frac{r-11.r-10.r-9.r-8.r-7.r-6.r-5}{5040},$$

$$C' = \frac{r-10.r-9.r-8.r-7.r-6.r-5}{720},$$

$$C'' = \frac{r-9.r-8.r-7.r-6.r-5}{120},$$

$$C''' = \frac{r-8.r-7.r-6.r-5}{24},$$

$$C^{\text{iv}} = \frac{r-7.r-6.r-5}{6},$$

and thence $C = \frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040}$.

I omit the calculation, as the equation is at once seen to be a particular case of a known factorial formula.

25. We may analyse the partitions of an r-gon into a given number of parts, according to the nature of the parts, that is the numbers of the sides of the several component polygons. It is for this purpose convenient to introduce the notion of "weight"; say a triangle has *the* weight 1, then a quadrangle, as divisible into two triangles, has

the weight 2, a pentagon, as divisible into three triangles, has the weight 3, ... and generally an r-gon, as divisible into $r-2$ triangles, has the weight $r-2$. It at once follows that, if

$$W = w+w', \text{ or } = w+w'+w'', \text{ \&c.},$$

then a polygon of weight W is divisible into two polygons of the weights w, w', or into three polygons of the weights w, w', w'' respectively; and so on. Thus the 2-partitions of an 8-gon (weight $= 6$) are 15, 24, and 33; the 3-partitions are 114, 123, 222, and so on. Of course the number of the partitions 15, 24, 33, is equal to the whole number of the 2-partitions of the 8-gon, that is $= 20$; the number of the partitions 114, 123, 222, is equal to the whole number of the 3-partitions of the 8-gon, that is it is $= 120$; and so in other cases. It is easy to derive in order one from the other the numbers of the partitions of each several kind of the polygons of the several weights 2, 3, 4, 5, 6, &c.; and I write down the accompanying Table (No. 26), facing page 260, the process for the construction being as follows:

27. The first column (2 parts) is at once obtained. For a polygon of an odd number of sides, for instance the 9-gon (weight $= 7$), imagining the summits numbered in order 1, 2, ... 9, we divide this into a triangle and octagon, or obtain the partitions 16, by drawing the diagonals 13, 24, ... 81, 92: viz., the number is $= 9$. In the Table this is written, $16 = 9$; and so in other cases. Similarly we divide it into a quadrangle and heptagon, or obtain the partitions 25, by drawing the diagonals 14, 25, ... 82, 93: viz., the number is again $= 9$; and we divide it into a pentagon and a hexagon, or obtain the partitions 34, by drawing the diagonals 15, 26, ... 83, 94: viz., the number is $= 9$, and here

$$9+9+9 = 27,$$

the whole number of 2-partitions of the 9-gon. For a polygon of an even number of sides, for instance the 10-gon (weight $= 8$), the process is a similar one, the only difference being that for the division into two hexagons (that is, for the partitions 44) each partition is thus obtained twice, or the number of such partitions is $\frac{1}{2}10, = 5$; the numbers for the partitions 17, 26, 35, 44, thus are 10, 10, 10, 5; and we have

$$10+10+10+5 = 35,$$

the whole number of the 2-partitions of the 10-gon.

28. To obtain the second column (3 parts)—suppose, for instance, the 3-partitions of the 9-gon; these are 115, 124, 133, 233. We obtain

the number of the partitions 115 from the terms

$$16 = 9 \quad \text{and} \quad 25 = 9$$

of the first column: viz., in 16, changing the 6 into 15, that is, dividing the polygon of weight 6 into two parts of weights 1 and 5 respectively; this can be done in eight ways (see, higher up, 15 = 8 in the first column), and we thus obtain the number of partitions

$$9 \times 8 = 72;$$

and again, in 25, changing the 2 into 11, that is, dividing the polygon of weight 2 into two parts each of weight 1; this can be done in two ways (see, higher up, 11 = 2 in the first column), and we thus obtain the number of partitions

$$9 \times 2 = 18;$$

we should thus have, for the number of partitions 115, the sum

$$72 + 18 = 90,$$

only, as it is easy to see, each partition is obtained twice, and the number of the partitions 115 is the half of this, = 45. And by the like process it is found that the numbers of the partitions 124, 133, 233 are equal to 90, 45, 45 respectively; and then, as a verification we have

$$45 + 90 + 45 + 45 = 225,$$

the whole number of the 3-partitions of the 9-gon.

29. The third column (4 parts) is derived in like manner from the second column by aid of the first column; and so in general each column is derived in like manner from the column which immediately precedes it, by aid of the first column. And we have for the numbers in each compartment of any column the verification that the sum of these numbers is equal to the whole number (for the proper values of k and r) of the k-partitions of the r-gon.

It might be possible, by an application of the method of generating functions, to find a law for the numbers in any compartment of a column of the table; but I have not attempted to make this investigation.

30. In the table in No. 2, the numbers 1, 2, 5, 14, 42, &c., of the diagonal line show the number of partitions of the triangle, the quadrangle, the 5-gon, ... r-gon into triangles: viz., these numbers show the number of partitions of the r-gon into $r-2$ parts, that is, into triangles; and, for the r-gon, writing

$$k = r - 2,$$

of ways in which this can be done is $P_aP_bP_c$; and it is to be noticed of ways in which this can be done is $P_a P_b P_c$; and it is to be noticed

the number is
$$=\frac{[2r-4]^{r-3}}{[r-2]^{r-3}}.$$

If, as above, taking the weight of the triangle to be 1, we write
$$r-2=w,$$
then the number is
$$=\frac{[2w]^{w-1}}{[w]^{w-1}},$$
viz., this is the expression for the number of partitions of the polygon of weight w, or $(w+2)$-gon, into triangles.

31. The question considered by Taylor and Rowe, in the paper referred to in No. 1, is that of the partition of the r-gon into p-gons, for p, a given number > 3; this implies a restriction on the form of r, viz., we must have $r-2$ divisible by $p-2$. In fact (generalizing the definition of w), if we attribute to a p-gon the weight 1, and accordingly to a polygon divisible into w p-gons the weight w, then, r being the number of summits, we must have
$$r=(p-2)\,w+2.$$
In particular, if $p=4$, or the r-gon is to be divided into quadrangles, then r is necessarily even, and for the values
$$w=1,\ 2,\ 3,\ \ldots,$$
we have
$$r=4,\ 6,\ 8,\ \ldots.$$

32. To fix the ideas, I assume $p=4$, or consider the problem of the division of the $(2w+2)$-gon into quadrangles. Writing
$$w-1=a+b+c,$$
we take at pleasure any one side of the $(2w+2)$-gon, making this the first side of a quadrangle, the second, third, and fourth sides being diagonals of the polygon such that outside the second side we have a polygon of weight a, outside the third side a polygon of weight b, and outside the fourth side a polygon of weight c. Any one or more of the numbers a, b, c may be $=0$ (they cannot be each of them $=0$ except in the case $w=1$). The meaning is that the corresponding side of the quadrangle, instead of being a diagonal, is a side of the $(2w+2)$-gon, viz., there is no polygon outside such side. Suppose in general that P_w is the number of ways in which a polygon of weight w can be divided in quadrangles, and let each of the polygons of weights a, b, c respectively, be divided into quadrangles, the number of ways in which this can be done is $P_aP_bP_c$; and it is to be noticed

that, if for instance $a = 0$, then the number is $= P_b P_c$, viz., the formula remains true if only we assume $P_0 = 1$. The number of partitions thus obtained is $\Sigma P_a P_b P_c$, where the summation extends to all the partitions of $w-1$ into the parts a, b, c (zeros admissible and the order of the parts being attended to). And we thus obtain *all* the partitions of the $(2w+2)$-gon into w parts; for first the partitions so obtained are all distinct from each other, and next every partition of the $(2w+2)$-gon into w parts is a partition in which the selected side of the $(2w+2)$-gon is a side of some one of the quadrangles. That is,

we have $\qquad P_w = \Sigma P_a P_b P_c$ (a, b, c as above);

and it hence appears that, considering the generating function

$$f = 1 + P_1 x + P_2 x^2 + P_3 x^3 + \ldots,$$

we have

$$f = 1 + x f^3.$$

The reasoning is precisely the same if, instead of a division into quadrangles, we have a division into p-gons; the only difference is that instead of the three parts a, b, c, we have the $p-1$ parts a, b, c ..., and the equation for f thus is

$$f = 1 + x f^{p-1}.$$

33. Writing for a moment

$$f = u + x f^{p-1},$$

and expanding by Lagrange's theorem, we have

$$f = u + \frac{x}{1}(u^{p-1}) + \frac{x^2}{1.2}(u^{2(p-1)})' \ldots + \frac{x^w}{1.2\ldots w}(u^{w(p-1)})^{\prime\ldots(w-1)} + \ldots,$$

viz., after the differentiation, writing $u = 1$, we have

$$P_w = \frac{[(p-1)\,w]^{w-1}}{[w]^{w-1}},$$

where it will be recollected that for the number of sides of the polygon,

we have $\qquad r = (p-2)\,w + 2.$

In the case of the partition into triangles, $p = 3$, and we have the before mentioned value

$$[2w]^{w-1} \div [w]^{w-1}, \quad w = r - 2.$$

Thursday, April 9th, 1891.

Major P. A. MACMAHON, R.A., F.R.S., Vice-President, in the Chair.

Mr. J. Rose-Innes, B.Sc. London, B.A. Camb., was elected a member.

The following communication was made:—

> The Analytical Forms called Trees, with Applications to the Combinations of certain Electrical Quantities, and to the Compositions of Multipartite Numbers: Major Macmahon. (Mr. Walker in the Chair.)

A discussion followed, in which Messrs. Kempe, Hammond, and S. Roberts took part.

Mr. Kempe spoke on the flaw in his *proof* "On the Map-colour Theorem," which had recently been detected by Mr. P. J. Heawood, and showed that a statement by the latter at the close of his paper failed.* He further stated that the subject had long engaged his attention, and that he was still unable to solve the question to his satisfaction.

Mr. Tucker communicated a paper by Mr. Culverwell, "Compounded Solutions in the Calculus of Variations."

The following presents were received:—

"Educational Times," for April.

"Transactions of the Cambridge Philosophical Society," Vol. xv., Part I.

"Proceedings of the Cambridge Philosophical Society," Vol. VII., Part III.

"Œuvres Complètes de Christiaan Huygens," Vol. III., 4to; La Haye, 1890.

"Bulletin des Sciences Mathématiques," Tome xv., Feb., 1891.

"Bulletin de la Société Mathématique de France," Tome xix., Nos. 1 and 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 5 and 6, Roma, 1891.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stück 3; Leipzig, 1891.

"Rendiconti del Circolo Matematico di Palermo," Tomo v., Fasc. I. e II.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 125 and 126, and two parts of Index.

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* *Cf. Proc. Lond. Math. Soc.*, Appendix to Vol. xxi.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1890, III. and IV.; Leipzig, 1891.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. x., No. 1.

"Jahrbuch über die Fortschritte der Mathematik," Band xx., Heft 2; Jahrgang 1888, Heft 2.

A number of pamphlets by Dr. L. Kronecker, viz.—

> Offprints from the "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1888, Art. xvi., xviii.; 1889, Art. vi., x., xiv., xviii., xix., xxx., xxxi., xlii.; 1890, Art. vi., vii., xiv., xvi., xxvi., xxviii., xxx., xxxvi., xl., xlviii., liii.; 1891, Art. ii., iii. From "Crelle's Journal," Band c., Heft 4; ci., 4; cii., 3; civ., 4; cv., 2. And "Festschrift.... Mathematischen Gesellschaft in Hamburg 1890" (Uber die Dirichletsche Methode der Werthestimmung der Gaussschen Reihen); Leipzig, 1890.

Also a number by M. A. Mannheim, viz.—

> Offprints from "Rendiconti del Circolo Matematico," Tomo III. (1889, 10 Marzo).
>
> "Comptes Rendus," 10 Fév., 24 Fév., 3 Nov., 1 Dec., 1890; 2 Fév., 2 Mars, 1891.
>
> "Journal de l'École Polytechnique," LXe cahier, 1890.

Stability of Orbits. *By* Mr. A. G. GREENHILL.

[*Read May* 10*th*, 1888.]

1. The discussion of the stability of a circular orbit described under a central force is given in Section IX., Book I., of Newton's *Principia*; and it is there shown that if the force is proportional to a single power of the distance, represented by μr^{n-3}, the apsidal angle in the circular orbit, when slightly disturbed, is $\pi/n^{\frac{1}{2}}$; so that n must be positive for the apsidal angle to be real, and for the orbit to be stable.

Representing as usual the reciprocal of r by u, and the central force P by μu^m, then for the circular orbit to be stable, m must not *be greater* than 3.

This is easily seen from a comparison of the velocity in the circular orbit with the velocity acquired under the attraction from an infinite distance; for

$$\frac{(\text{vel.})^2 \text{ in circle}}{(\text{vel.})^2 \text{ from infinity}} = \frac{Pr}{\int_r^\infty P\,dr} = \frac{m-1}{2}.$$

So that, if $m = 3$, the velocity in the circular orbit is equal to the velocity from infinity; if $m > 3$, the velocity in the circle is greater than the velocity from infinity, and the body if slightly disturbed can recede to an infinite distance; but, if $m < 3$, the body if slightly disturbed will not be able to recede to any finite extent, and the disturbed orbit will cut the circular orbit at regular intervals, and consequently be a stable orbit, *e.g.*, the orbit of the Moon relatively to the Sun.

When $m = 2$, the orbit is of course a conic section, with a focus at the centre of force, and the apsidal angle is π; and when $m = -1$, the orbit is an ellipse, with its centre at the centre of force, and the apsidal angle is π.

2. In a circular orbit of radius a, described with mean angular motion n, under a force to the centre varying as the m^{th} power of the distance, we may put

$$P = n^2 a\,(r/a)^m;$$

or expanding in powers of $(r-a)/a$, and, neglecting squares and higher powers,

$$P = n^2 a + mn^2\,(r-a);$$

so that, if

$$P = \mu + \nu\,(r-a),$$

$m = \nu/n^2$, and the apsidal angle is

$$\pi/(3+m)^{\frac{1}{2}} = \pi/(3+\nu/n^2)^{\frac{1}{2}}.$$

And, generally, if we have $P = \text{f}r$, and expand P in powers of $r-a$,

$$P = \text{f}a + (r-a)\,\text{f}'a + \ldots,$$

and the apsidal angle is

$$\pi/(3 + a\text{f}'a/\text{f}a)^{\frac{1}{2}}.$$

3. As an illustration, consider a particle, weighing m grammes, describing a circle of radius a centims. on a smooth horizontal table, with mean motion n, by being attached to one end of a fine thread, which passes through a hole in the table at the centre of the circle, and supports another particle weighing m' grammes; so that

$$mn^2a = m'g \text{ (dynes)}.$$

Now, if slightly disturbed,

$$r^2 d\vartheta/dt = h = a^2n,$$

and, by the principle of energy,

$$\tfrac{1}{2}(m+m')\,\dot{r}^2+\tfrac{1}{2}mr^2\dot{\vartheta}^2-m'g\,(l-r) = \tfrac{1}{2}ma^2n^2-m'g\,(l-a)\,\text{(ergs)},$$

l denoting the length of the thread.

Therefore $$\tfrac{1}{2}(m+m')\frac{a^4n^2}{r^4}\,\frac{dr^2}{d\vartheta^2}+\tfrac{1}{2}m\,\frac{a^4n^2}{r^2}+mn^2ar = \tfrac{3}{2}ma^2n^2,$$

or $$\tfrac{1}{2}\,\frac{m+m'}{m}\,\frac{dr^2}{d\vartheta^2} = -\tfrac{1}{2}r^2\left(\frac{r}{a}-1\right)^2\left(2\,\frac{r}{a}+1\right)$$

(the only real solution of which is $r = a$, $dr/d\vartheta = 0$),

and $$\frac{m+m'}{m}\,\frac{d^2r}{d\vartheta^2} = -r\left(\frac{r}{a}-1\right)\left(5\,\frac{r^2}{a^2}-\frac{r}{a}-1\right);$$

and putting $$r/a = 1+\rho,$$

and neglecting squares and higher powers of ρ in the slightly disturbed orbit,

$$\frac{m+m'}{m}\,\frac{d^2\rho}{d\vartheta^2} = -3\rho,$$

so that the apsidal angle is $\pi\sqrt{\{(m+m')/3m\}}$

(Thomson and Tait, *Natural Philosophy*, §§ 319, 350).

4. As another similar illustration, take the case of a particle moving on the interior of a smooth vertical cone with vertex downwards.

With columnar cylindrical coordinates r, θ, z,

$$r^2 d\vartheta/dt = h,$$

and $$\tfrac{1}{2}\dot{r}^2+\tfrac{1}{2}r^2\dot{\vartheta}^2+\tfrac{1}{2}\dot{z}^2+gz = gz_0,$$

z_0 denoting the height above the vertex of the cone of the plane of *zero velocity.*

If α denote the semivertical angle of the cone,

$$r = z \tan \alpha\,;$$

and

$$\tfrac{1}{2}\dot{z}^2 \sec^2 \alpha + \tfrac{1}{2}\frac{h^2}{z^2}\cot^2 \alpha + gz = gz_0,$$

or $\tfrac{1}{2}z^2\dot{z}^2 \sec^2 \alpha = -gz^2(z-z_0) - \tfrac{1}{2}h^2 \cot^2 \alpha = -g(z-z_1)(z-z_2)(z-z_3)$,

suppose, z_1, z_2, denoting the level of the two circles between which the motion takes place, so that

$$z_0 > z_1 > z > z_2 > z_3\,;$$

and by the theory of equations,

$$z_1 + z_2 + z_3 = z_0,$$

$$z_2 z_3 + z_3 z_1 + z_1 z_2 = 0\,;$$

so that

$$z_3 = -\frac{z_1 z_2}{z_1 + z_2}, \quad z_0 = \frac{z_1^2 + z_1 z_2 + z_2^2}{z_1 + z_2}\,;$$

while

$$z_1 z_2 z_3 = -\tfrac{1}{2}h^2 (\cot \alpha)^2 / g.$$

Then

$$\ddot{z} \sec^2 \alpha = -g + \frac{h^2}{z^3}\cot^2 \alpha = -g\left(1 + \frac{2z_1 z_2 z_3}{z^3}\right);$$

so that

$$d^2z/dt^2 = 0,$$

when

$$z^3 = -2z_1 z_2 z_3 = \frac{2z_1^2 z_2^2}{z_1 + z_2}.$$

When $z_1 = z_2$, $\qquad z_0 = \tfrac{3}{2}z_1, \quad z_3 = -\tfrac{1}{2}z_1,$

and the particle will be describing a circle; and then

$$\tfrac{1}{2}z^2\dot{z}^2 \sec^2 \alpha = -g(z-z_1)^2(z + \tfrac{1}{2}z_1),$$

and

$$\ddot{z} \sec^2 \alpha = -g\left(1 - \frac{z_1^3}{z^3}\right);$$

so that, if

$$\frac{z}{z_1} = 1 + \rho,$$

where ρ is small, $\qquad \ddot{\rho}\, z_1 \sec^2 \alpha = -3g\rho,$

and the small oscillations will synchronize with a simple pendulum of length $\tfrac{1}{3}z_1 \sec^2 \alpha$.

In the projection of the motion on a horizontal plane

$$\tfrac{1}{2}\dot{r}^2 \operatorname{cosec}^2 a + \tfrac{1}{2}r^2\dot{\vartheta}^2 + g\,(r-r_0)\cot a = 0,$$

or
$$\tfrac{1}{2}h^2 \operatorname{cosec}^2 a\,\frac{dr^2}{d\vartheta^2} = -gr^4\,(r-r_0)\cot a - \tfrac{1}{2}h^2r^2$$

$$= -gr^2\cot a\,(r-r_1)(r-r_2)(r-r_3),$$

where
$$r_0 > r_1 > r > r_2 > r_3.$$

In the circular orbit, slightly disturbed, put

$$r_1 = r_2;$$

then
$$r_0 = \tfrac{3}{2}r_1, \quad r_3 = -\tfrac{1}{2}r_1;$$

and
$$\tfrac{1}{2}h^2 \operatorname{cosec}^2 a\,\frac{dr^2}{d\vartheta^2} = -gr^2\cot a\,(r-r_1)^2\,(r+\tfrac{1}{2}r_1),$$

while
$$h = nr_1^2, \quad g\cot a = n^2r_1;$$

so that
$$\tfrac{1}{2}r_1^3 \operatorname{cosec}^2 a\,\frac{dr^2}{d\vartheta^2} = -r^2\,(r-r_1)^2\,(r+\tfrac{1}{2}r_1);$$

and
$$r_1^3 \operatorname{cosec}^2 a\,\frac{d^2r}{d\vartheta^2} = -r\,(r-r_1)(5r^2-rr_1-r_1^2);$$

so that, putting $r/r_1 = 1+\sigma$ in the slightly disturbed circular orbit,

$$\operatorname{cosec}^2 a\,\frac{d^2\sigma}{d\vartheta^2} = -3\sigma,$$

and the apsidal angle is $\tfrac{1}{3}\pi\sqrt{3}\sin a$.

By changing the sign of g, we obtain the equations of motion of a particle on the outside of a vertical cone with vertex upwards; so that

$$\tfrac{1}{2}r_1^3 \operatorname{cosec}^2 a\,\frac{dr^2}{d\vartheta^2} = r^2\,(r-r_1)^2\,(r+\tfrac{1}{2}r_1),$$

and the motion in the circle is now unstable.

Integrating, we find for the unstable motion below the circle

$$\vartheta\sin a = \int_r^\infty \frac{r_1^{\frac{3}{2}}\,dr}{r\,(r-r_1)\sqrt{(2r+r_1)}}$$

$$= \int\left(\frac{1}{r-r_1} - \frac{1}{r}\right)\sqrt{\left(\frac{r_1}{2r+r_1}\right)}\,dr$$

$$= \frac{2}{\sqrt{3}}\cosh^{-1}\sqrt{\frac{2r+r_1}{2r-2r_1}} - 2\cosh^{-1}\sqrt{\frac{2r+r_1}{2r}}.$$

For the unstable motion above the circle

$$\vartheta \sin a = \int_0 \frac{r_1^{\frac{3}{2}}\, dr}{r\,(r_1-r)\sqrt{(r_1+2r)}}$$

$$= \int \left(\frac{1}{r_1-r}+\frac{1}{r}\right)\sqrt{\left(\frac{r_1}{r_1+2r}\right)}\, dr$$

$$= \frac{2}{\sqrt{3}}\sinh^{-1}\sqrt{\frac{r_1+2r}{2r_1-2r}}+2\cosh^{-1}\sqrt{\frac{r_1+2r}{2r}}$$

+an infinite constant.

The apsidal angle for a very elongated orbit has been investigated by Whewell (*Trans. Camb. Phil. Soc.*); he shows that the apsidal angle is the $1/\sqrt{(3-n)}$ of the apsidal angle in a very nearly circular orbit.

5. If $n = 3$, the apsidal angle given above becomes infinite, and in this separating case the orbits are Cotes's spirals, the equations of which are, by gradually increasing the velocity at given distance from the origin, included in the forms

(i.) $r \cosh m\vartheta = a$;

(ii.) $r \exp m\vartheta = a$, the equiangular spiral; the orbit in which the velocity is that due to infinite distance; so that a body, slightly disturbed from the circular orbit, recedes in an equiangular spiral either to an infinite distance, or to the origin;

(iii.) $r \sinh m\vartheta = a$;

(iv.) $r m\vartheta = a$, the reciprocal spiral, and then dr/dt is constant;

(v.) $r \sin m\vartheta = a$. (Cayley, *British Association Report*, 1862.)

For values of n greater than 3, the circular orbit becomes unstable, and it will be interesting to investigate the orbits for $n = 4, 5, 6, 7, \ldots$ which the body will be found ultimately describing, when slightly displaced from the circular orbit.

6. Denoting by a the radius of the circular orbit, then (i.), for $n = 5$, the body will be found ultimately describing the orbit

$$au = \tanh \tfrac{1}{2}\sqrt{2}\vartheta, \quad r = a \coth \tfrac{1}{2}\sqrt{2}\vartheta,$$

outside the circle, or

$$au = \coth \tfrac{1}{2}\sqrt{2}\vartheta, \quad r = a \tanh \tfrac{1}{2}\sqrt{2}\vartheta,$$

inside the circle, according to the direction of the initial slight disturbance; both orbits having the asymptotic circle $au = 1$, $r = a$, corresponding to $\vartheta = \infty$, an infinite number of revolutions.

To prove this, we have from the ordinary differential equation of motion

$$\frac{d^2u}{d\vartheta^2} + u = \frac{P}{h^2u^2},$$

with

$$P = \mu u^5,$$

$$\frac{d^2u}{d\vartheta^2} + u = \frac{\mu}{h^2}u^3;$$

also, in the circular orbit,

$$h^2 = v^2a^2 = n^2a^4 = Pa^3 = \mu/a^2,$$

or

$$\mu/h^2 = a^2;$$

so that, putting $au = x$,

$$\frac{d^2x}{d\vartheta^2} + x = x^3.$$

Integrating,

$$\tfrac{1}{2}\left(\frac{dx^2}{d\vartheta^2} + x^2\right) = \tfrac{1}{4}x^4 + C,$$

and C is determined from the condition that

$$\frac{dx}{d\vartheta} = 0, \quad \text{where} \quad x = 1;$$

therefore

$$C = \tfrac{1}{4},$$

and

$$\frac{dx^2}{d\vartheta^2} = \tfrac{1}{2}x^4 - x^2 + \tfrac{1}{2} = \tfrac{1}{2}(x^2-1)^2,$$

$$\frac{dx}{d\vartheta} = \pm\tfrac{1}{2}\sqrt{2}\,(x^2-1).$$

For the orbit outside the circle, since $x^2 < 1$, take

$$\frac{dx}{d\vartheta} = \tfrac{1}{2}\sqrt{2}\,(1-x^2),$$

$$\tfrac{1}{2}\sqrt{2}\,d\vartheta = \frac{dx}{1-x^2},$$

$$\tfrac{1}{2}\sqrt{2}\,\vartheta = \int\frac{dx}{1-x^2} = \tanh^{-1}x$$

or

$$x = \tanh\tfrac{1}{2}\sqrt{2}\,\vartheta, \quad r = a\coth\tfrac{1}{2}\sqrt{2}\,\vartheta.$$

Similarly, for the inside orbit

$$\tfrac{1}{2}\sqrt{2}\,\vartheta = \int^{\infty} \frac{dx}{x^2-1} = \coth^{-1}x;$$

or

$$x = \coth \tfrac{1}{2}\sqrt{2}\,\vartheta, \quad r = a \tanh \tfrac{1}{2}\sqrt{2}\,\vartheta.$$

7. With $n = 4$, we shall find the body ultimately describing either

$$au = \frac{\cosh\vartheta - 2}{\cosh\vartheta + 1}, \quad r = a\,\frac{\cosh\vartheta + 1}{\cosh\vartheta - 2} \text{ (the outside orbit)};$$

or

$$au = \frac{\cosh\vartheta + 2}{\cosh\vartheta - 1}, \quad r = a\,\frac{\cosh\vartheta - 1}{\cosh\vartheta + 2} \text{ (the inside orbit)}.$$

For, now

$$\frac{d^2u}{d\vartheta} + u = \frac{\mu}{h^2}\,u^2,$$

and

$$h^2 = \mu/a,$$

so that $(au = a)$

$$\frac{d^2x}{d\vartheta^2} + x = x^2;$$

$$\tfrac{1}{2}\left(\frac{dx^2}{d\vartheta^2} + x^2\right) = \tfrac{1}{3}x^3 + C = \tfrac{1}{3}x^3 + \tfrac{1}{6},$$

subject to the condition that $dx/d\vartheta = 0$, when $x = 1$;

$$\frac{dx^2}{d\vartheta^2} = \tfrac{2}{3}x^3 - x^2 + \tfrac{1}{3} = \tfrac{1}{3}(x-1)^2(2x+1),$$

$$\frac{dx}{d\vartheta} = \pm\tfrac{1}{3}\sqrt{3}\,(x-1)\,\sqrt{(2x+1)}.$$

In the outside orbit, $x < 1$,

$$\frac{d\vartheta}{dx} = \frac{\sqrt{3}}{(1-x)\sqrt{(1+2x)}},$$

$$\vartheta = \int \frac{\sqrt{3}\,dx}{(1-x)\sqrt{(1+2x)}} = 2\sinh^{-1}\sqrt{\frac{1+2x}{2-2x}}$$

$$= 2\cosh^{-1}\sqrt{\frac{3}{2-2x}} = \cosh^{-1}\frac{2+x}{1-x};$$

so that

$$\cosh\vartheta = \frac{2+x}{1-x},$$

or

$$au = x = \frac{\cosh\vartheta - 2}{\cosh\vartheta + 1}.$$

In the inner orbit, $x > 1$,

$$\vartheta = \int \frac{\sqrt{3}\, dx}{(x-1)\sqrt{(2x+1)}}$$

$$= 2\cosh^{-1}\sqrt{\frac{2x+1}{2x-2}} = 2\sinh^{-1}\sqrt{\frac{3}{2x-2}}$$

$$= \cosh^{-1}\frac{x+2}{x-1},$$

$$\cosh\vartheta = \frac{x+2}{x-1},$$

$$au = x = \frac{\cosh\vartheta+2}{\cosh\vartheta-1}.$$

8. When $P = \mu u^7$,

$$\frac{d^2u}{d\vartheta^2} + u = \frac{\mu}{h^2}u^5,$$

or

$$\frac{d^2x}{d\vartheta^2} + x = x^5,$$

$$\tfrac{1}{2}\left(\frac{dx^2}{d\vartheta^2} + x^2\right) = \tfrac{1}{6}x^6 + C = \tfrac{1}{6}x^6 + \tfrac{1}{3},$$

if subject to the condition that $dx/d\vartheta = 0$ when $x = 1$;

$$\frac{dx^2}{d\vartheta^2} = \tfrac{1}{3}(x^6 - 3x^2 + 2) = \tfrac{1}{3}(x^2-1)^2(x^2+2),$$

$$\sqrt{3}\frac{dx}{d\vartheta} = \pm(x^2-1)\sqrt{(x^2+2)}.$$

For the outside orbit, $x^2 < 1$,

$$\frac{d\vartheta}{dx} = \frac{\sqrt{3}}{(1-x^2)\sqrt{(2+x^2)}},$$

$$\vartheta = \cosh^{-1}\sqrt{\frac{2+x^2}{2-2x^2}} = \sinh^{-1}\sqrt{\frac{3x^2}{2-2x^2}}$$

$$= \tfrac{1}{2}\cosh^{-1}\frac{1+2x^2}{1-x^2},$$

$$a^2u^2 = x^2 = \frac{\cosh 2\vartheta - 1}{\cosh 2\vartheta + 2}, \quad r^2 = a^2\frac{\cosh 2\vartheta + 2}{\cosh 2\vartheta - 1}.$$

For the inner orbit, $x^2 > 1$,

$$\vartheta = \int \frac{\sqrt{3}\,dx}{(x^2-1)\sqrt{(x^2+2)}} = \tfrac{1}{2}\cosh^{-1}\sqrt{\frac{2x^2+1}{x^2-1}},$$

or
$$a^2u^2 = \frac{\cosh 2\vartheta+1}{\cosh 2\vartheta-2}, \quad r^2 = a^2\frac{\cosh 2\vartheta-2}{\cosh 2\vartheta+1}.$$

9. For $n=6$, $P=\mu u^6$, we find

$$\frac{d^2x}{d\vartheta^2}+x = x^4,$$

$$\tfrac{1}{2}\left(\frac{dx^2}{d\vartheta^2}+x^2\right) = \tfrac{1}{5}x^5+\tfrac{3}{10},$$

$$\frac{dx^2}{d\vartheta^2} = \tfrac{2}{5}x^5-x^2+\tfrac{3}{5} = \tfrac{1}{5}(x-1)^2(2x^3+4x^2+6x+3),$$

so that elliptic functions will be required in the equation of this orbit.

We reduce this integral to the Weierstrassian form by putting

$$x = \tfrac{1}{3}(s+k),$$

and choosing k so that $\quad 2x^3+4x^2+6x+3$

$$= \tfrac{2}{27}(s+k)^3+\tfrac{4}{9}(s+k)^2+2(s+k)+3$$

$$= \tfrac{1}{54}\left\{4(s+k)^3+24(s+k)^2+108(s+k)+162\right\}$$

$$= \tfrac{1}{54}(4s^3-g_2s-g_3),$$

the term in s^2 being made to disappear; and therefore

$$k=-2,$$

and then
$$g_2=-60, \quad g_3=-10,$$

$$\Delta = g_2^3-27g_3^2 = -218700,$$

$$J = \frac{g_2^3}{\Delta} = \frac{80}{81}, \quad J-1 = \frac{1}{81}.$$

Then
$$\frac{d\vartheta}{ds} = \frac{3\sqrt{30}}{\pm(s-5)\sqrt{(4s^3+60s+10)}}$$

the upper or lower sign being taken, according as the disturbed orbit is inside or outside the unstable circular orbit.

Denoting by e_1, e_2, e_3 the roots of the cubic

$$S = 4s^3 + 60s + 10 = 0,$$

then the real root $\quad e_2 = -\tfrac{1}{2}(\sqrt[3]{100} - 2\sqrt[3]{10}),$

while $\quad e_1 = -\tfrac{1}{2}(\omega\sqrt[3]{100} - 2\omega^2\sqrt[3]{10}),$

$$e_3 = -\tfrac{1}{2}(\omega^2\sqrt[3]{100} - 2\omega\sqrt[3]{10}),$$

ω, ω^2 denoting the imaginary cube roots of unity.

Again, putting $\quad c = 16\kappa^2\kappa'^2,$

$$J = \frac{4(c-1)^3}{27c} = \frac{80}{81},$$

from which we find $\quad c = \tfrac{1}{3}(\sqrt[3]{100} + 2\sqrt[3]{10} + 3).$

Putting $\quad s = \wp u, \quad$ and $\quad \wp v = 5, \quad \wp' v = 9\sqrt{10},$

then $\quad \vartheta = \int \frac{3\sqrt{30}\,du}{\wp u - \wp v} = \frac{1}{\sqrt{3}}\int \frac{\wp' v\,du}{\wp u - \wp v} = \frac{1}{\sqrt{3}}\log\frac{\sigma(u-v)}{\sigma(u+v)}e^{u\zeta v},$

or $$e^{\sqrt{3}\vartheta} = \frac{\sigma(u-v)}{\sigma(u+v)}e^{u\zeta v},$$

the equation of the orbit.

I have not yet been able to verify a conjecture that v is an aliquot part of the real period ω_2 of these elliptic functions.

10. With $\quad n = 8, \quad P = \mu u^8,$

we obtain, with the same notation as before,

$$\frac{d^2x}{d\vartheta^2} + x = x^6,$$

$$\tfrac{1}{2}\frac{dx^2}{d\vartheta^2} + \tfrac{1}{2}x^2 = \tfrac{1}{7}x^7 + \tfrac{5}{14},$$

in the unstable orbit, in which

$$dx/d\vartheta = 0,$$

when $$x = 1;$$

and then $\quad \frac{dx^2}{d\vartheta^2} = \tfrac{1}{7}(2x^7 - 7x^2 + 5)$

$$= \tfrac{1}{7}(x-1)^2(2x^5 + 4x^4 + 6x^3 + 8x^4 + 10x + 5),$$

so that the corresponding integral for the orbit is hyperelliptic.

8. Generally for any even value of *n*, say

$$P = \mu u^{2m},$$

in the circular orbit, $$\frac{d^2x}{d\vartheta^2} + x = x^{2m-2},$$

and integrating, $$\frac{dx^2}{d\vartheta^2} = \frac{2x^{2m-1}}{2m-1} - x^2 + 1 - \frac{2}{2m-1},$$

or $$(2m-1)\frac{dx^2}{d\vartheta^2}$$

$$= (x-1)^2 \left\{2x^{2m-3} + 4x^{2m-5} + 6x^{2m-7} + \ldots + (4m-6)\,x + 2m-3\right\}.$$

The equation is simpler for an odd value of *n*; for, if

$$P = \mu u^{2m+1},$$

$$\frac{d^2x}{d\vartheta^2} + x = x^{2m-1},$$

$$\frac{dx^2}{d\vartheta^2} = \frac{x^{2m}}{m} - x^2 + 1 - \frac{1}{m},$$

or $$m\frac{dx^2}{d\vartheta^2} = (x^2-1)^2 \left\{x^{2m-4} + 2x^{2m-6} + 3x^{2m-8} + \ldots + (m-2)\,x^2 + m-1\right\};$$

a function of x^2 only.

The cases of $m = 1, 2, 3$ have already received consideration.

11. With $$P = \mu u^9,$$

$$\frac{dx^2}{d\vartheta^2} = \tfrac{1}{4}(x^2-1)^2(x^4+2x^2+3),$$

and $$\vartheta = \int \frac{\pm 2dx}{(x^2-1)\sqrt{(x^4+2x^2+3)}}.$$

To reduce this elliptic integral to the Legendrian and Jacobian form, we put

$$\frac{x^2}{\sqrt{3}} = \frac{1-z}{1+z},$$

then $$x^4+2x^2+3 = \frac{6+2\sqrt{3}+(6-2\sqrt{3})\,z^2}{(1+z)^2} = 12\frac{\kappa'^2+\kappa^2z^2}{(1+z)^2},$$

with $$\kappa^2 = \frac{\sqrt{3}-1}{2\sqrt{3}}, \quad \kappa'^2 = \frac{\sqrt{3}+1}{2\sqrt{3}}, \quad 4\kappa^2\kappa'^2 = \tfrac{2}{3};$$

and $$\frac{x\,dx}{\sqrt{3}} = -\frac{dz}{(1+z)^2},$$

$$dx = -\sqrt[4]{3}\,\frac{dz}{(1+z)^2}\sqrt{\frac{1+z}{1-z}};$$

so that $$\sqrt[4]{3}\,\vartheta = \int \frac{\pm dz}{(x^2-1)\sqrt{(1-z^2\,.\,\kappa'^2+\kappa^2z^2)}} = \int \frac{\pm d\,\mathrm{cn}^{-1}z}{x^2-1}.$$

Putting
$$z = \operatorname{cn} u,$$
so that
$$x^2 = \sqrt{3}\,\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u};$$
then
$$\sqrt[4]{3}\,\vartheta = \int \frac{\pm du}{\left(\sqrt{3}\,\dfrac{1-\operatorname{cn} u}{1+\operatorname{cn} u}-1\right)} = \int \frac{\pm(1+\operatorname{cn} u)\,du}{\sqrt{3}-1-(\sqrt{3}+1)\operatorname{cn} u}$$
$$= \frac{\pm 1}{\sqrt{3}+1}\int\left(1+\frac{3-\sqrt{3}}{\operatorname{cn} u-2+\sqrt{3}}\right)du,$$

leading to elliptic integrals of the third kind; and the upper or lower sign is to be taken, according as the inner or outer orbit to the unstable circle is considered.

If β denotes the modular angle of these elliptic functions,
$$\sin^2 2\beta = 4\kappa^2\kappa'^2 = \tfrac{2}{3},$$
$$\sin 2\beta = \sqrt{\tfrac{2}{3}}, \quad \cos 2\beta = \kappa'^2-\kappa^2 = \sqrt{\tfrac{1}{3}}.$$

If we put
$$2-\sqrt{3} = \operatorname{cn} v,$$
then
$$\operatorname{sn} v = \sqrt[4]{3}\,(\sqrt{3}-1), \quad \operatorname{dn} v = \frac{3-\sqrt{3}}{\sqrt{2}},$$
and
$$\sqrt[4]{3}\,\vartheta = \pm\left\{\tfrac{1}{2}(\sqrt{3}-1)\,u+\frac{1}{\sqrt{2}\,\sqrt[4]{3}}\int\frac{\operatorname{sn} v\operatorname{dn} v\,du}{\operatorname{cn} u-\operatorname{cn} v}\right\};$$

and here again it is reasonable to conjecture that v is likely to be an aliquot part of the real period K.

12. In the reduction of these integrals to the Weierstrassian form, we put
$$x^2 = \tfrac{2}{3}(s-1);$$
then
$$x^4+2x^2+3 = \tfrac{1}{9}(2s+1)^2+2,$$
and
$$\frac{dx}{\sqrt{(x^4+2x^2+3)}} = \frac{3\sqrt{6}\,ds}{\sqrt{S}},$$
where
$$S = 4s^3+15s-19,$$
so that
$$g_2 = -15, \quad g_3 = 19, \quad \Delta = -2\times 3^8,$$
and
$$J = \frac{g_2^3}{\Delta} = \frac{5^3}{2\times 3^5}, \quad J-1 = -\frac{19^2}{2\times 3^5}.$$

Then in the inner orbit, in which x, and therefore s, is ultimately infinite, we put

$$s = \wp u,$$

so that
$$\vartheta = \int \frac{\frac{3}{2}\sqrt{6}\,du}{\wp u - \frac{5}{2}},$$

and if we put
$$\wp v = \tfrac{5}{2},$$

then
$$\wp' v = -9,$$

and
$$\sqrt{6}\,\vartheta = \int \frac{-\wp' v\,du}{\wp u - \wp v},$$

a canonical integral of the *third kind*, in Weierstrass's notation, so that

$$\sqrt{6}\,\vartheta = \log \frac{\sigma(v+u)}{\sigma(v-u)}\,e^{-u\zeta v}.$$

13. Another substitution to employ in the reduction to the Weierstrassian form is

$$\sqrt{(x^4+2x^2+3)} + x^2 + \tfrac{1}{3} = \tfrac{2}{3}t,$$

so that x and t are infinite together as in the inner orbit; and then

$$x^2 = \frac{4t^2-4t-26}{12(t+1)} = \frac{4t^3-g_2't-g_3'}{12(t+1)^2},$$

with
$$g_2' = 30, \quad g_3' = 26, \quad \Delta' = 2^2 \times 3^7,$$

and
$$J' = \frac{2 \times 5^3}{3^4}.$$

But this value of J' is derived from the preceding value of J by means of Klein's quadric transformation

$$J = \frac{(4\tau-1)^3}{27\tau}, \quad J' = \frac{(4\tau'-1)^3}{27\tau'}, \quad \tau\tau' = 1,$$

by taking
$$\tau = \tfrac{2}{3}, \quad \tau' = \tfrac{3}{2}.$$

With this second substitution,

$$x^2+1 = \frac{4t^2+8t-14}{12(t+1)},$$

$$x^4+2x^2+3 = (x^2+1)^2+2 = \left\{\frac{2t^2+4t+11}{6(t+1)}\right\}^2;$$

and
$$T = 4t^3 - g_2' t - g_3 = 12x^2 (t+1)^2,$$

$$\sqrt{T} = 2\sqrt{3}\, x\, (t+1).$$

Also
$$2x \frac{dx}{dt} = \frac{2t^2+4t+11}{6\,(t+1)^2} = \frac{\sqrt{(x^4+2x^2+3)}}{t+1},$$

or
$$\frac{dx}{\sqrt{(x^4+2x^2+3)}} = \frac{dt}{2x\,(t+1)} = \frac{\sqrt{3}\, dt}{\sqrt{T}}.$$

And
$$x^2 - 1 = \frac{2t^2-8t-19}{6\,(t+1)} = \frac{(t-2)^2 - \frac{27}{2}}{3\,(t+1)},$$

so that
$$\vartheta = \int^{\infty} \frac{2dx}{(x^2-1)\sqrt{(x^4+2x^2+3)}}$$

$$= \int \frac{6\sqrt{3}\,(t+1)\, dt}{\left\{(t-2)^2 - \frac{27}{2}\right\} \sqrt{T}},$$

and when
$$x^2 = 1,$$

$$t = 2 \pm \tfrac{3}{2}\sqrt{6}, \quad \sqrt{T} = 6\sqrt{3} \pm 9\sqrt{2}.$$

Putting
$$s = \wp u, \quad \sqrt{S} = -\wp' u;$$

and
$$\wp v_1 = 2 + \tfrac{3}{2}\sqrt{6}, \quad \wp' v_1 = 6\sqrt{3} + 9\sqrt{2},$$

$$\wp v_2 = 2 - \tfrac{3}{2}\sqrt{6}, \quad \wp' v_2 = 6\sqrt{3} - 9\sqrt{2},$$

then
$$\sqrt{6}\,\vartheta = \int \left(\frac{\wp' v_1}{\wp u - \wp v_1} + \frac{\wp' v_2}{\wp u - \wp v_2} \right) dv,$$

the sum of two elliptic integrals of the third kind.

14. The above is the substitution for the inner orbit, in which $x^2 > 1$; but for tho outer orbit the corresponding substitution is

$$\tfrac{2}{3} t = \frac{\sqrt{3}\,\sqrt{(3+2x^2+x^4)} + 3 + \tfrac{1}{3}x^2}{x^2},$$

making t infinite when $x = 0$.

Then
$$x^2 = \frac{18\,(t+1)}{2t^2-2t-13} = \frac{36\,(t+1)^2}{4t^3-30t-26},$$

$$x^2 + 1 = \frac{2t^2+16t+5}{2t^2-2t-13},$$

$$3+2x^2+x^4=(1+x^2)^2+2=3\left(\frac{2t^2+4t+11}{2t^2-2t-13}\right)^2,$$

$$1-x^2=\frac{2t^2-20t-31}{2t^2-2t-13},$$

so that now

$$\vartheta=\int_0 \frac{2dx}{(1-x^2)\sqrt{(3+2x^2+x^4)}}=\int^\infty \frac{18\sqrt{3}\,(t+1)\,dt}{(2t^2-20t-31)\sqrt{T}},$$

reducible as above to the sum of two elliptic integrals of the third kind.

15. These two last substitutions are particular cases of the general theory of the reduction of differential elliptic element $dx/\sqrt{X}$, where

$$X=ax^4+4bx^3+6cx^2+4dx+e,$$

as quoted by Biermann, in *Problemata quædam mechanica functionum ellipticarum ope soluta*, from Weierstrass's lectures.

The substitution is

$$s=\left(\frac{\sqrt{X}+\sqrt{Y}}{x-y}\right)^2-\tfrac{1}{4}a\,(x+y)^2-b\,(x+y)-c,$$

so that s replaces the arbitrary constant in Euler's integral of the equation

$$dx/\sqrt{X}=dy/\sqrt{Y};$$

and now each of them is equal to $-ds/\sqrt{S}$, where

$$S=4s^3-g_2s-g_3,$$

$$g_2=ae-4bd+3c^2,\quad g_3=ace+2bcd-ad^2-eb^2-c^3;$$

so that now
$$v-w=\int_y^x \frac{dx}{\sqrt{X}}=\int_s^\infty \frac{ds}{\sqrt{S}},$$

and
$$s=\wp\,(v-w).$$

In our application, we havé

$$a=1,\quad b=0,\quad c=\tfrac{1}{3},\quad d=0,\quad e=3,$$

so that
$$g_2=\frac{10}{3},\quad g_3=\frac{26}{27};$$

and in the inner orbit we take y infinite, and in the outer orbit we make y zero.

16. Again, for $P = \mu u^{11}$,

$$5\frac{dx^2}{d\vartheta^2} = (x^2-1)^2(x^6+2x^4+3x^2+4)\,;$$

and this is reduced to the Weierstrassian form by putting

$$x^2 = \frac{4}{s+k}\,;$$

when

$$\frac{dx}{\sqrt{(x^6+2x^4+3x^2+4)}} = -\frac{\frac{1}{2}ds}{\sqrt{\{4(s+k)^3+12(s+k)^2+32(s+k)+64\}}}$$

$$= -\frac{\frac{1}{2}ds}{\sqrt{(4s^3-g_2s-g_3)}},$$

by choosing $k = -1$;

and then $g_2 = -20, \quad g_3 = -40$;

$$J = \frac{5}{2^5}, \quad J-1 = -\frac{3^3}{2^5}.$$

Now

$$1-x^2 = \frac{s-5}{s-1},$$

$$\frac{1}{1-x^2} = 1+\frac{4}{s-5}\,;$$

so that

$$\pm\frac{\vartheta}{\sqrt{5}} = \int\left(1+\frac{4}{s-5}\right)\frac{\frac{1}{2}ds}{\sqrt{S}},$$

an elliptic integral of the third kind.

If we put $s = \wp u$,

and $\wp v = 5$,

then $\wp' v = 4\sqrt{10}$;

and the integral is reduced to Weierstrass's canonical form.

17. To illustrate the case of a stable orbit, take

$$n = 2, \quad P = \mu u^2\,;$$

then in the circular orbit

$$\frac{dx^2}{d\vartheta^2} = -(1-x)^2,$$

the only solution of which is

$$x = 1, \quad \frac{dx}{d\vartheta} = 0\,;$$

the orbit being stable.

For an orbit slightly disturbed, we must have

$$\frac{dx^2}{d\vartheta^2} = e^2 - (1-x)^2,$$

where e is a small constant; and if we assume

$$(1-x)^2 = e^2 \cos^2 n\vartheta,$$

then

$$\frac{dx^2}{d\vartheta^2} = e^2 \sin^2 n\vartheta\,;$$

and therefore $n = 1$, and the apsidal angle is π; also

$$au = x = 1 - e\cos\vartheta,$$

the polar equation of a conic section; all well known results.

18. We notice the square factor $(x-1)^2$ and another factor which is always positive, in the value of $dx^2/d\vartheta^2$ in these unstable orbits; the vanishing of the square factor corresponding to the unstable circular orbit.

In all these orbits, in which $n > 3$, the circle

$$r = a, \quad\text{or}\quad au = 1,$$

is an asymptotic circle of the orbit, and the circle is attained or departed from appreciably only after an infinite number of revolutions.

In the discussion then of the case in a central orbit, where it is supposed that at a certain point the centre of curvature of the orbit coincides with the centre of force, and in answer to the question as to what takes place, whether the body will move in the orbit or the circle of curvature, we notice now that such a point can only be attained asymptotically after an infinite number of revolutions.

For a stable case of the circular orbit, the centre of curvature cannot coincide with the centre of force, unless the orbit is circular to begin with; thus, for example, when $n = 2$, the centre of curvature of an ellipse cannot coincide with a focus, unless the ellipse is a circle.

19. For a single centre of force the simplest orbit is the circle, and we consider the stability or instability of the circular orbit, under a single central force to the centre.

But given, for instance, Euler's problem of the two centres of force attracting or repelling according to the Newtonian law, then confocal ellipses or hyperbolas with foci at the centres of force will be the simplest orbits we can consider in relation to stability, the elliptic coordinates ϕ or ϑ, defined by

$$x + iy = c\cosh(\phi - i\vartheta),$$

being constant along these orbits.

But with polar coordinates, the degenerate case where

$$x + iy = c\exp(\phi + i\vartheta),$$

then ϕ or r is constant along the concentric circular orbits, and ϑ along the rectilinear radial orbits.

The corresponding equations of motion have been investigated in an article in the *Proceedings of the London Mathematical Society*, Vol. xi., No. 162, and it will be interesting to examine this investigation in its relation to the question of stability, as thereby a new classification of the different orbits can be obtained.

20. First, however, let us determine the conditions under which an ellipse or hyperbola, or portion of an ellipse or hyperbola, can be described freely.

By a well-known theorem (Sylvester's "Astronomical Prolusions," *Phil. Mag.*, 1866), if one and the same orbit can be described freely under different forces, it can be described under all these forces acting simultaneously, with velocity equal to the square root of the sum of the squares of the separate velocities; and the direction of any force can be reversed, if we treat the corresponding squared velocity as negative.

Familiar instances occur with confocal ellipses, which can be described under forces to the foci S, S', varying inversely as the square of the distance, the forces acting separately, or in conjunction, and we may also add a force to the centre O, varying as the distance.

By a proposition in Frost's *Newton*, Prop. VI., Theorem V., the pressure on a particle in a smooth elliptic tube, moving under forces to the foci S, S', varying inversely as the square of the distance, is inversely proportional to the radius of curvature; but this pressure is always zero, and the particle will therefore describe the ellipse freely without constraint, if the forces are equal at the point where the velocity is zero, one force being attractive and the other repulsive.

Consequently a particle, placed anywhere upon the curve (a circle) where the forces are equal, will, starting from rest, proceed to oscillate in the arc of an ellipse.

Suppose, for instance, the forces are of equal strength, that is, equal at equal distances; then a particle placed anywhere upon the line OB bisecting SS' at right angles, will proceed to oscillate on the arc BAB', of an ellipse, having S and S' as foci, if the force to S attracts, and from S' repels.

But if both attract or both repel, the body will either oscillate in the straight line BOB', or will be driven off from B in a straight line to infinity.

Also, if projected from any point in the line joining SS', at right angles to SS', with the velocity from infinity, the particle will describe a hyperbola.

Generally, if the forces at S, S' are not of equal strength, but both attract, or both repel, a particle, placed anywhere on the circle where the forces are equal, will oscillate in the finite arc of a hyperbola, if both forces attract, but will move off to infinity on the hyperbola, if both repel.

Thus, if the same hyperbola is described by a particle under an attraction to one focus, or a repulsion from the other focus, the velocities are equal where the forces are equal.

Two attracting spheres at S, S' may be considered as the centres of force at S, S'; but if one force is repulsion, it may be supposed that a solid homogeneous sphere, whose centre is at S, has had excavated an excentric cavity, with centre at S', provided the strength of the repulsion is less than that of the attraction.

If A denotes the mass (in grammes) of the solid homogeneous sphere, and B the mass excavated, C the constant of gravitation, r, s the distances of a point P from S, S', then a particle projected from P in the direction bisecting the angle SPS', with appropriate velocity, can be made to describe a hyperbola.

By bringing S and S' close together to O so as to make the cavity only slightly excentric, we may imitate the action of a molecule at O, which has a potential partly gravitational and partly magnetic.

If the potential is entirely magnetic, a particle, placed anywhere on OB, will oscillate on a semicircle BAB', with centre O.

We proceed now to investigate the stability of this kind of motion.

21. *Motion under two Newtonian Centres of Force.*

Euler, "Mémoires de Berlin," 1760;
Lagrange, "Mécanique Analytique," t. ii., Chap. III.;
Legendre, "Fonctions Elliptiques," t. i., p. 411;

Jacobi, "Lectures on Dynamics," *Gesammelte Werke;*

Königsberger, "Dissertatio de motu puncti versus duo fixa centra attracti," Berolini, 1869;

Cayley, "Collected Works," Vol. III., *Phil. Mag.*, 1858; *Brit. Assoc. Report*, 1862;

Sylvester, "Astronomical Prolusions," *Phil. Mag.*, 1866.

We begin by putting

$$x+iy = c\cosh(\phi+i\vartheta),$$

so that $x = c\cosh\phi\cos\vartheta,\quad y = c\sinh\phi\sin\vartheta;$

and we use ϑ, ϕ as the variable coordinates; and now if ϕ is constant and ϑ varies, the particle describes an ellipse with fixed foci S, S' at $(\pm c, 0)$, and semiaxes $c\cosh\phi$, $c\sinh\phi$, and excentricity $\operatorname{sech}\phi$; and ϑ is the excentric angle or excentric anomaly of the particle.

Conversely, if ϑ is constant, and ϕ varies, the particle describes a confocal hyperbola, of semiaxes $c\cos\vartheta$, $c\sin\vartheta$, and excentricity $\sec\vartheta$; and the variable ϕ may, by analogy, be called the hyperbolic excentric anomaly.

By putting $\vartheta = nt$ or $\phi = nt$, we obtain ellipses and hyperbolas described under a central force to the origin, varying as the distances.

Now, using s_1 and s_2 to denote the arcs of the ellipse and confocal hyperbola, we have

$$\frac{ds_1^2}{d\vartheta^2} = \frac{\partial x^2}{\partial\vartheta^2}+\frac{\partial y^2}{\partial\vartheta^2} = c^2(\cosh^2\phi\sin^2\vartheta+\sinh^2\phi\cos^2\vartheta)$$

$$= c^2(\cosh^2\phi-\cos^2\vartheta);$$

and $$\frac{ds_2^2}{d\phi^2} = \frac{\partial x^2}{\partial\phi^2}+\frac{\partial y^2}{\partial\phi^2},$$

has the same value, namely the Jacobian

$$J = \frac{d(x, y)}{d(\phi, \vartheta)}.$$

For we may put

$$x+iy = z,\quad x-iy = z',\quad \phi+i\vartheta = w,\quad \phi-i\vartheta = w',$$

so that $$z = c\cosh w,\quad \frac{dz}{dw} = c\sinh w;$$

and then, by the theory of conjugate functions,

$$\begin{aligned} J &= c^2\sinh w\sinh w' = c^2\sinh(\phi+i\vartheta)\sinh(\phi-i\vartheta) \\ &= \tfrac{1}{2}c^2(\cosh 2\phi-\cosh 2i\vartheta) = \tfrac{1}{2}c^2(\cosh 2\phi-\cos 2\vartheta) \\ &= c^2(\cosh^2\phi-\cos^2\vartheta). \end{aligned}$$

22. Now, for a particle of unit mass (one gramme), the kinetic energy (in ergs.) is

$$T = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}(\dot{s}_1^2+\dot{s}_2^2)$$

$$= \tfrac{1}{2}c^2(\cosh^2\phi-\cos^2\vartheta)(\dot{\vartheta}^2+\dot{\varphi}^2) = J(\tfrac{1}{2}\dot{\vartheta}^2+\tfrac{1}{2}\dot{\varphi}^2).$$

Denoting by r and s the distances (in centimetres) of a point from the foci ($\pm c$, 0), then

$$s+r = 2c\cosh\varphi, \quad s-r = 2c\cos\vartheta,$$

or
$$r = c(\cosh\phi-\cos\vartheta), \quad s = c(\cosh\varphi+\cos\vartheta);$$

and now denoting by A, B, or rather by Ac^3/C, Bc^3/C the attracting masses in grammes, or by Ac^3, Bc^3 the strength of the Newtonian centres of force, C denoting the *gravitation constant* in C.G.S. units, then the gravitation potential (in ergs.)

$$U = Ac^3r^{-1}+Bc^3s^{-1} = \frac{Ac^2}{\cosh\phi-\cos\vartheta}+\frac{Bc^2}{\cosh\varphi+\cos\vartheta}.$$

We may introduce centres of force at O, S, S', attracting as the distance, by adding terms $\alpha r+\beta s$ to U; and we may suppose the particle projected so as to describe a tortuous curve in space; but these generalities do not materially alter the solution.

With $\alpha+\beta = 0$, the additional field of force is a uniform field, like that due to gravity, if SS' is vertical.

23. The Lagrangian equations of motion are now

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\vartheta}}-\frac{\partial T}{\partial\vartheta} = \frac{\partial U}{\partial\vartheta} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (1),$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}}-\frac{\partial T}{\partial\phi} = \frac{\partial U}{\partial\phi} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (2);$$

leading, with our coordinates, to

$$(\cosh^2\phi-\cos^2\vartheta)\,\ddot{\vartheta}+\sin\vartheta\cos\vartheta\,(\dot{\vartheta}^2-\dot{\phi}^2)+\sinh\phi\cosh\phi\,2\dot{\vartheta}\dot{\phi}$$

$$= \frac{\partial V}{c^2\partial\vartheta} = -\frac{A\sin\vartheta}{(\cosh\phi-\cos\vartheta)^2}+\frac{B\sin\vartheta}{(\cosh\varphi+\cos\vartheta)^2}\ldots\ldots\ldots(3),$$

$$(\cosh^2\phi-\cos^2\vartheta)\,\ddot{\varphi}+\sin\vartheta\cos\vartheta\,.\,2\dot{\vartheta}\dot{\phi}-\sinh\phi\cosh\phi\,(\dot{\vartheta}^2-\dot{\phi}^2)$$

$$= \frac{\partial V}{\partial\phi} = -\frac{A\sinh\phi}{(\cosh\phi-\cos\vartheta)^2}-\frac{B\sinh\phi}{(\cosh\varphi+\cos\vartheta)^2}\;\ldots\ldots\;(4).$$

Multiplying (3) by $\dot{\vartheta}$, (4) by $\dot{\phi}$, adding and integrating, we obtain

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)(\dot{\vartheta}^2+\dot{\phi}^2)=\frac{A}{\cosh\phi-\cos\vartheta}+\frac{B}{\cosh\phi+\cos\vartheta}+H,$$

where H is an arbitrary constant; or

$$T=U+Hc^2 \text{..............................} (5),$$

the equation of energy.

24. To obtain Euler's second integral of equations (3) and (4), we multiply (3) by $\sinh^2\phi\,.\,\dot{\vartheta}$, (4) by $\sin^2\vartheta\,.\,\dot{\phi}$, subtract and integrate; and then

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)(\sinh^2\phi\,\dot{\vartheta}^2-\sin^2\vartheta\,\dot{\phi}^2)$$

$$=\int\left(\frac{\partial U}{\partial\vartheta}\sinh^2\phi\,d\vartheta-\frac{\partial U}{\partial\phi}\sin^2\vartheta\,d\phi\right)+K=V+K \text{............} (6),$$

where K is an arbitrary constant, and V is a function such that

$$\frac{\partial V}{\partial\vartheta}=\frac{\partial U}{c^2\partial\vartheta}\sinh^2\phi,\quad \frac{\partial V}{\partial\phi}=-\frac{\partial U}{c^2\partial\phi}\sinh^2\vartheta,$$

and then we readily find that

$$V=A\,\frac{\cos\vartheta\cosh\phi-1}{\cosh\phi-\cos\vartheta}-B\,\frac{\cos\vartheta\cosh\phi+1}{\cosh\phi+\cos\vartheta} \text{............} (7),$$

so that $\qquad V=A\cos ASP-B\cos AS'P,$

where ASP, $AS'P$ are the true anomalies of the point P from the perihelion A about S and the aphelion A about S'.

25. From equations (5) and (6), we find

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\,\dot{\vartheta}^2=(Uc^{-2}+H)\sin^2\vartheta+V+K$$

$$=(A-B)\cos\vartheta-H\cos^2\vartheta+H+K \text{......} (8),$$

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\,\dot{\phi}^2=(Uc^{-2}+H)\sinh^2\phi-V-K$$

$$=(A+B)\cosh\phi+H\cosh^2\phi-H-K \text{......} (9),$$

so that, by division,

$$\frac{d\vartheta^2}{d\phi^2}=\frac{(A-B)\cos\vartheta-H\cos^2\vartheta+H+K}{(A+B)\cosh\phi+H\cosh^2\phi-H-K} \text{............} (10),$$

the differential equation of an orbit, an equation in which the variables are separated.

Euler employs new variables u and v, which will be found to be such that

$$u = \tan \tfrac{1}{2}\vartheta, \quad v = \tanh \tfrac{1}{2}\phi,$$

Euler's p and q being $c \cos \vartheta$ and $c \cosh \phi$.

But in Legendre's notation (*Fonctions Elliptiques*, t. I., p. 412) q and p are used to denote $\tan \frac{1}{2}\vartheta$ and $\tanh \frac{1}{2}\phi$, and the true anomalies ASP, $AS'P$ from perihelion about S and aphelion about S' are given by

$$\tan \tfrac{1}{2}ASP = q/p, \quad \tan \tfrac{1}{2}AS'P = pq.$$

We shall employ Legendre's notation

$$q = \tan \tfrac{1}{2}\vartheta, \quad p = \tanh \tfrac{1}{2}\phi;$$

and now

$$\frac{d\vartheta}{dq} = \frac{2}{1+q^2}, \quad \frac{d\phi}{dp} = \frac{2}{1-p^2};$$

$$\cos \vartheta = \frac{1-q^2}{1+q^2}, \quad \cosh \phi = \frac{1+p^2}{1-p^2};$$

so that equation (10) becomes transformed to

$$\frac{dp^2}{dq^2} = \frac{P}{Q},$$

or

$$\pm \frac{dp}{\sqrt{P}} = \frac{dq}{\sqrt{Q}},$$

(Legendre, *F. E.*, t. I., p. 413), where

$$P = (A+B)(1-p^4) + H(1+p^2)^2 - (H+K)(1-p^2)^2,$$

$$Q = (A-B)(1-q^4) - H(1-q^2)^2 + (H+K)(1+q^2)^2,$$

or

$$P = (A+B)(1-p^4) + 4Hp^2 - K(1-p^2)^2,$$

$$Q = (A-B)(1-q^4) + 4Hq^2 + K(1+q^2)^2,$$

so that our H and K are the $2C$ and C' of Legendre.

The functions P and Q being of the fourth degree in p and q, the differential relation,

$$dp^2/P = dq^2/Q = dR^2,$$

gives p and q as elliptic functions of the variable parameter R; and by the elimination of R we obtain the equations of an orbit as a relation between p and q, and thence between ϑ and ϕ.

26. To obtain the time t in the orbit, we have, from equations (8) and (9),

$$\frac{dt^2}{(\cosh^2\phi-\cos^2\vartheta)^2}=\frac{\frac{1}{2}d\vartheta^2}{(A-B)\cos\vartheta-H\cos^2\vartheta+H+K}=\frac{dq^2}{Q}$$

$$=\frac{\frac{1}{2}d\phi^2}{(A+B)\cosh\phi+H\cosh^2\phi-H-K}=\frac{dp^2}{P}$$

$$=dR^2\text{, suppose;}$$

$$dt=(\cosh^2\phi-\cos^2\vartheta)\,dR=(\sinh^2\phi+\sin^2\vartheta)\,dR;$$

$$t=\pm\int\left(\frac{1+p^2}{1-p^2}\right)^2\frac{dp}{\surd P}-\int\left(\frac{1-q^2}{1+q^2}\right)^2\frac{dq}{\surd Q}$$

$$=\pm\int\frac{4p^2}{(1-p^2)^2}\frac{dp}{\surd P}+\int\frac{4q^2}{(1+q^2)^2}\frac{dq}{\surd Q},$$

assuming that dq/dt is positive at the start, or that the excentric anomaly ϑ is increasing.

27. Legendre devotes 125 more pages to the discusssion and classification of the different cases of the orbits which may arise.

Let us begin by considering the stability of an elliptic orbit about two centres of force in the foci, both attracting.

Considered as moving in the ellipse with velocities v and v' under the centres of force taken singly; then

$$\tfrac{1}{2}v^2=\frac{Ac^3}{r}-\frac{Ac^3}{2a},$$

$$\tfrac{1}{2}v'^2=\frac{Bc^3}{s}-\frac{Bc^3}{2a};$$

and therefore in the orbit under the forces combined, we have the velocity V given by

$$\tfrac{1}{2}V^2=\tfrac{1}{2}(v^2+v'^2)=\tfrac{1}{2}c^2(\cosh^2\phi-\cos^2\vartheta)(\dot\vartheta^2+\dot\phi^2)$$

$$=\frac{Ac^3}{r}+\frac{Bc^3}{s}-\frac{A+B}{2a}c^3,$$

so that $\quad H=-(A+B)\,c/2a=-\tfrac{1}{2}(A+B)\operatorname{sech}\beta,$

if $\quad a=c\cosh\beta$

is the semi-major axis of the elliptic orbit.

Also $d\phi/dt = 0$ when $\phi = \beta$, so that

$$(A+B)\cosh\beta + H\cosh^2\beta - H - K = 0,$$

or

$$H+K = \tfrac{1}{2}(A+B)\cosh\beta;$$

and now

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)^2\dot{\phi}^2 = -\tfrac{1}{2}(A+B)(\cosh\phi - \cosh\beta)^2\operatorname{sech}\beta;$$

and the negative sign shows that this elliptic orbit is stable, the only real solution being

$$\phi = \beta, \quad \dot{\phi} = 0.$$

At the same time

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)^2\dot{\vartheta}^2 = (A-B)\cos^2\vartheta - H\cos^2\vartheta + H + K$$

$$= \tfrac{1}{2}A(\cosh\beta + \cos\vartheta)^2\operatorname{sech}\beta + \tfrac{1}{2}B(\cosh\beta - \cos\vartheta)^2\operatorname{sech}\beta;$$

or, with

$$\phi = \beta,$$

$$\frac{d\vartheta^2}{dt^2} = \frac{A\operatorname{sech}\beta}{(\cosh\beta - \cos\vartheta)^2} + \frac{B\operatorname{sech}\beta}{(\cosh\beta + \cos\vartheta)^2},$$

$$\frac{dt^2}{d\vartheta^2} = \frac{\cosh\beta}{A(\cosh\beta - \cos\vartheta)^{-2} + B(\cosh\beta + \cos\vartheta)^{-2}}.$$

With

$$\cos\vartheta = \frac{1-q^2}{1+q^2}, \qquad d\vartheta = \frac{2dq}{1+q^2};$$

$$\cosh\phi = \frac{1+p^2}{1-p^2}, \quad \cosh\beta = \frac{1+b^2}{1-b^2},$$

we find this changes to

$$\frac{dt^2}{dq^2} = 16\,\frac{1+b^2}{(1-b^2)^3}\,\frac{(b^2+q^2)^2(1+b^2q^2)^2}{A(1+b^2q^2)^2 + B(b^2+q^2)^2},$$

which, on putting $z = \dfrac{b^2+q^2}{1+b^2q^2}$, becomes

$$\frac{dt^2}{dz^2} = 4\,\frac{(1+b^2)^3}{1-b^2}\,\frac{z^2(z-b^2)}{(1-b^2z)(A+Bz^2)},$$

$$\frac{dt}{dz} = 2(1+b^2)\sqrt{\frac{1+b^2}{1-b^2}}\,\frac{z(z-b^2)}{\sqrt{\{(z-b^2)(1-b^2z)(A+Bz^2)\}}},$$

giving t as an elliptic integral of the second kind.

28. We notice that the condition that the elliptic orbit $\phi = \beta$ is described, is expressed by the appearance of the squared factor $(\cosh\phi - \cosh\beta)^2$ in the expression of $(d\phi/dt)^2$; and this condition is

of universal application, even when only a portion of the ellipse is described; and when a hyperbola or a portion of a hyperbola is described, the corresponding condition will be expressed by the presence of the squared factor $(\cos\vartheta-\cos\alpha)^2$ in $(d\vartheta/dt)^2$ for the hyperbola $\vartheta=\alpha$.

29. Suppose the centre of force at S' becomes repulsive, we change the sign of B, and now

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\vartheta}=(A+B)\cos\vartheta-H\cos^2\vartheta+H+K,$$

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\phi}^2=(A-B)\cosh\phi+H\cosh^2\phi-H-K;$$

and if the ellipse $\phi=\beta$, or a portion of it, is described, then

$$(\cosh^2\phi-\cos^2\vartheta)^2\dot{\phi}^2=-(A-B)(\cosh\beta-\cosh\phi)^2\operatorname{sech}\beta,$$

since

$$H=-\tfrac{1}{2}(A-B)\operatorname{sech}\beta,$$

$$H+K=\tfrac{1}{2}(A-B)\cosh\beta;$$

and

$$(\cosh^2\phi-\cos^2\vartheta)^2\dot{\vartheta}^2$$

$$=A(\cosh\beta+\cos\vartheta)^2\operatorname{sech}\beta-B(\cosh\beta-\cos\vartheta)^2\operatorname{sech}\beta.$$

Now $d\vartheta/dt=0$, and the velocity is therefore zero, where $\vartheta=\alpha$, so that

$$A(\cosh\beta+\cos\alpha)^2-B(\cosh\beta-\cos\alpha)^2=0,$$

or

$$Ar^{-2}=Bs^{-2},$$

that is, where the forces are equal, as we have already proved.

We shall find it convenient to represent the equal values of these forces at the point (α,β) where the particle is placed at rest, and then allowed to move under the attractions, by Mc^{-2}, so that

$$A=M(\cosh\beta-\cos\alpha)^2,$$

$$B=M(\cosh\beta+\cos\alpha)^2;$$

and now

$$\tfrac{1}{2}(A+B)=M(\cosh^2\beta+\cos^2\alpha),$$

$$\tfrac{1}{2}(A-B)=-2M\cos\alpha\cosh\beta;$$

$$H=M\cos\alpha,$$

$$H+K=-M\cos\alpha\cosh\beta,$$

$$K=-M\cos\alpha(\cosh^2\beta+1);$$

and

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\phi}^2 = 2M\cos\alpha\,(\cosh\beta-\cosh\phi)^2,$$

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\vartheta}^2 = 2M(\cos\vartheta-\cos\alpha)(\cosh^2\beta-\cos\alpha\cos\vartheta);$$

so that

$$\frac{\pm\,d\phi}{\cosh\beta-\cosh\phi} = \frac{\surd(\cos\alpha)\,d\vartheta}{\surd\{(\cos\vartheta-\cos\alpha)(\cosh^2\beta-\cos\alpha\cos\vartheta)\}}$$

is the differential equation of the orbit; but, unless $\cos\alpha$ is positive, the only solution is $\phi=\beta$.

30. If B is greater than A, the particle P oscillates on the smaller half of the ellipse, and now the positive sign in the equation

$$\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)\,\dot{\phi}^2 = \tfrac{1}{2}(B-A)(\cosh\beta-\cosh\phi)^2\operatorname{sech}\beta,$$

shows that this orbit is *unstable*.

In this case $\cos\alpha$ is positive; and now, putting

$$\cos\vartheta=\frac{1-q^2}{1+q^2},\quad \cos\alpha=\frac{1-a^2}{1+a^2},$$

$$\cosh\phi=\frac{1+p^2}{1-p^2},\quad \cosh\beta=\frac{1+b^2}{1-b^2},$$

we obtain

$$\frac{\pm\,dp}{b^2-p^2}=\frac{\surd(1-a^4)\,dq}{\surd[(a^2-q^2)\{a^2+2b^2+a^2b^2+(1+2a^2b^2+b^4)\,q^2\}]}.$$

Putting $q=a\cos\chi$, we find that

$$\frac{\pm\,dp}{b^2-p^2}=\sqrt{\frac{1-a^2}{1+a^2}}\,\frac{1}{1+b^2}\,\frac{d\chi}{\surd(1-\kappa^2\sin^2\chi)},$$

or $$d\genfrac{}{}{0pt}{}{\tanh^{-1}}{\coth^{-1}}\frac{p}{b}=\frac{\cos^2\frac{1}{2}\alpha}{\surd(\cosh\beta)}\,d\operatorname{cn}^{-1}\frac{q}{a},$$

where $$\kappa^2=\frac{1+2a^2b^2+b^4}{(1+a^2)(1+b^2)^2}.$$

The upper sign gives the disturbed orbit tending towards the centre of the system; and then

$$\frac{1}{b}\,d\tanh^{-1}\frac{p}{b}=\frac{\cos^2\frac{1}{2}\alpha}{\surd(\cosh\beta)}\,d\operatorname{cn}^{-1}\frac{q}{a},$$

or $$\tanh^{-1}\frac{p}{b}=\frac{\sin\alpha}{2\surd\cosh\beta}\operatorname{cn}^{-1}\frac{q}{a}+C,$$

is the equation of the disturbed orbit, in which q therefore oscillates between $\pm a$, or ϑ between $\pm \alpha$.

Similarly, in the outer disturbed orbit, reaching to infinity,

$$\coth^{-1}\frac{p}{b} = \frac{\sin\alpha}{2\sqrt{(\cosh\beta)}}\,\mathrm{cn}^{-1}\frac{q}{a} + C.$$

With B less than A, the particle P oscillates on the larger half of the ellipse, and the negative sign in

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)^2\dot{\phi}^2 = -\tfrac{1}{2}(A-B)(\cosh\beta - \cosh\phi)^2\operatorname{sech}\beta$$

shows that the orbit is *stable*.

Now, the only solution is $\phi = \beta$; and then

$$2\sqrt{M}t = \int\frac{\cosh^2\beta - \cos^2\vartheta}{\sqrt{\{(\cos\vartheta - \cos\alpha)(\cosh^2\beta - \cos\alpha\cos\vartheta)\}}}\,d\vartheta,$$

which gives ϑ as an elliptic function of t, by elliptic integrals of the first and second kind.

When $A = B$, the particle, if placed anywhere at rest on the conjugate axis, will proceed to oscillate in the confocal ellipse; and now

$$\cos\alpha = 0,\quad H = 0,\quad K = 0,\quad \dot{\phi} = 0,\quad \phi = \beta;$$

and $$2\sqrt{M}t = \int\frac{\cosh^2\beta - \cos^2\vartheta}{\sqrt{(\cos\vartheta)}}\,d\vartheta.$$

31. Generally, when $H = 0$, the velocity is that due to a fall from infinity; and now, with two attracting centres

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)\,\dot{\vartheta}^2 = (A-B)\cos\vartheta + K,$$

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)\,\dot{\phi}^2 = (A+B)\cosh\phi - K;$$

and changing the sign of B changes the corresponding centre of force to one of repulsion.

The differential equation of the orbit is now

$$\frac{dp^2}{(1-p^2)\{A+B-K+(A+B+K)p^2\}}$$
$$= \frac{dq^2}{(1+q^2)\{A-B+K-(A-B-K)q^2\}}.$$

Suppose, in addition, that $A = B$; then

$$\frac{dq^2}{(1+q^2)^2} = \frac{K\,dp^2}{(1-p^2)\{2A-K+(2A+K)p^2\}};$$

so that $\tan^{-1} q$ is an elliptic integral of p, the form of the integral depending on the constant K; and simplifications occur when

$$K = \pm 2A.$$

32. Suppose, for instance, that the orbit cuts the line SS' beyond S or S' at right angles; then $\dot{\phi} = 0$, when $\phi = \beta$, and $\vartheta = 0$; and then

$$K = 2A \cosh \beta;$$

but, as this condition is independent of the value of ϑ, it merely expresses the fact that the orbit touches the ellipse $\phi = \beta$.

Then
$$\frac{dq^2}{(1+q^2)^2} = \frac{\cosh \beta \, dp^2}{(1-p^2)(2 \cosh^2 \tfrac{1}{2}\beta \, p^2 - 2 \sinh^2 \tfrac{1}{2}\beta)},$$

$$d \tan^{-1} q = \tfrac{1}{2}\lambda \, d\,(\mathrm{dn}^{-1} p),$$

or
$$p = \mathrm{dn}\, \lambda\, (\vartheta - \alpha),$$

or
$$\tanh \tfrac{1}{2}\phi = \mathrm{dn}\, \lambda\, (\vartheta - \alpha);$$

and $\lambda\alpha = K$, if $\dot{\phi} = 0$ when $\vartheta = 0$, $\phi = \beta$; also

$$\kappa' = \tanh \tfrac{1}{2}\beta, \quad \kappa = \operatorname{sech} \tfrac{1}{2}\beta.$$

If the orbit, instead of reaching to infinity, touches the ellipse $\phi = \gamma$, then we shall find, for the orbit,

$$\tanh \tfrac{1}{2}\phi = \tanh \tfrac{1}{2}\gamma \, \mathrm{dn}\, \lambda\, (\vartheta - \alpha),$$

with
$$\kappa' = \tanh \tfrac{1}{2}\beta / \tanh \tfrac{1}{2}\gamma.$$

Generally, if ϕ lies between β and γ, so that $\gamma > \phi > \beta$, and $\dot{\phi} = 0$ when $\phi = \beta$; and when $\phi = \gamma$, then H is negative, and

$$\tfrac{1}{2} (\cosh^2 \phi - \cos^2 \vartheta)^2 \dot{\phi} = -H (\cosh \gamma - \cosh \phi)(\cosh \phi - \cosh \beta),$$

and then
$$A + B = -H (\cosh \beta + \cosh \gamma),$$

$$H + K = -H \cosh \beta \cosh \gamma;$$

$$\tfrac{1}{2} (\cosh^2 \phi - \cos^2 \vartheta)^2 \dot{\vartheta}^2 = 2A \cos \vartheta - H (\cosh \beta - \cos \vartheta)(\cos \gamma - \cos \vartheta),$$

so that an additional condition connecting A and H is required in the general case.

If this condition is $\dot{\phi} = 0$ when $\vartheta = 0$, we obtain the preceding result.

33. For a hyperbolic orbit, $\vartheta = \alpha$, described continually in the same direction, taking A as attracting and B as repelling,

$$\tfrac{1}{2}v^2 = \frac{Ac^3}{r} + \frac{Ac^2}{2\cos\alpha},$$

$$\tfrac{1}{2}v'^2 = -\frac{Bc^3}{r} + \frac{Bc^2}{2\cos\alpha};$$

so that, with $V^2 = v^2 + v'^2$,

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)(\dot{\vartheta}_2 + \dot{\phi}^2)$$

$$= \frac{A}{\cosh\phi - \cos\vartheta} - \frac{B}{\cosh\phi + \cos\vartheta} + \tfrac{1}{2}(A+B)\sec\alpha;$$

so that $H = \tfrac{1}{2}(A+B)\sec\alpha.$

Also, from the condition that $\dot{\vartheta} = 0$ when $\vartheta = \alpha$,

$$0 = (A+B)\cos\alpha - H\cos^2\alpha + H + K;$$

so that, now,

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)\,\dot{\vartheta}^2 = -\tfrac{1}{2}(A+B)(\cos\vartheta - \cos\alpha)^2\sec\alpha,$$

as we might have inferred directly; and the negative sign shows that this hyperbolic orbit, described with continuous motion, is stable.

34. When a finite portion of a hyperbolic orbit $\vartheta = \alpha$ is described under two attractive forces, we have

$$\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)^2\,\dot{\vartheta}^2 = (A-B)\cos\vartheta - H\cos^2\vartheta + H + K$$

$$= -\tfrac{1}{2}(A-B)(\cos\vartheta - \cos\alpha)^2\sec\alpha;$$

and now, with $\cos\alpha$ positive, we have B greater than A, since the attractive forces are equal at the point $\vartheta = \alpha$, $\phi = \beta$, where the particle P is instantaneously at rest; and then with

$$A = M(\cosh\beta - \cos\alpha)^2,$$

$$B = M(\cosh\beta + \cos\alpha)^2,$$

$$\tfrac{1}{2}(A-B) = -2M\cos\alpha\cosh\beta,$$

we have $\tfrac{1}{2}(\cosh^2\phi - \cos^2\vartheta)\,\dot{\vartheta}^2 = M\cosh\beta\,(\cos\vartheta - \cos\alpha)^2,$

so that this finite orbit is unstable.

With the above values of H and K, viz.,

$$H = \tfrac{1}{2}(A-B)\sec\alpha, \quad H+K = -\tfrac{1}{2}(A-B)\cos\alpha,$$

$$\begin{aligned}\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\phi}^2 &= (A+B)\cosh\phi + H\cosh^2\phi - H - K \\ &= 2A\cosh\phi + \tfrac{1}{2}(A-B)(\cosh\phi-\cos\alpha)^2\sec\alpha \\ &= 2M(\cosh\beta-\cosh\phi)(\cosh\beta\cosh\phi-\cos^2\alpha);\end{aligned}$$

and now a calculation, similar to that already given, will lead to the relation connecting $q = \tan\tfrac{1}{2}\vartheta$, and an elliptic function of $p = \tanh\tfrac{1}{2}\phi$.

35. If, with $A=B$, the particle is projected from any point between S and S' perpendicularly to SS', or normally to any ellipse given by $\phi=\beta$, with velocity from infinity, then $\dot{\vartheta}=0$, when $\vartheta=\alpha$; so that $K=0$; and then $\dot{\vartheta}=0$ always, and the particle will proceed to describe the hyperbola $\vartheta=\alpha$; and then

$$2\sqrt{A}\,t = \int\frac{\cosh^2\phi-\cos^2\alpha}{\sqrt{(\cosh\phi)}}\,d\phi,$$

an elliptic integral of the second kind.

36. When, under a repulsion from both centres of force, the particle P, placed at the point where the forces of repulsion are equal, is driven off on the infinite arc of the hyperbola $\vartheta=\alpha$, starting from $\phi=\beta$, then

$$\begin{aligned}\tfrac{1}{2}(\cosh^2\phi-\cos^2\vartheta)^2\dot{\phi} &= (B-A)\cos\vartheta - H\cos^2\vartheta + H + K \\ &= -\tfrac{1}{2}(B-A)(\cos\vartheta-\cos\alpha)^2\sec\alpha,\end{aligned}$$

and this infinite hyperbolic orbit is therefore stable, since B is greater than A, if $\cos\alpha$ is positive.

In the complicated discussion of the different cases that arise in the general orbit, the classification is thus simplified by grouping the different kinds of orbits round these portions of elliptic and hyperbolic orbits, each as a central particular case.

37. Cartesian ovals are given by the relations

$$lr + ms = \text{constant},$$

or

$$L\cos\vartheta + M\cosh\phi = \text{constant};$$

and the forces requisite to make a particle move in a Cartesian oval can be investigated in this manner. (Sylvester, *Phil. Mag.*, 1866.)

38. When the foci S, S' are brought close together, the potential of A and B, if of opposite signs, will be partly gravitational and partly magnetic, and the potential may therefore be written

$$U = \mu r^{-1} + \nu r^{-2} \cos \vartheta,$$

if r, ϑ are the polar coordinates of P.

The original elliptic coordinates ϕ, ϑ now degenerate, so that

$$x + iy = c \exp (\phi + i\vartheta),$$

and ϑ thus becomes the vectorial angle of polar coordinates, while

$$r = ce^{\phi}.$$

Writing u for $1/r$, then

$$U = \mu u + \nu u^2 \cos \vartheta,$$

and the equations of motion, in polar coordinates, are

$$\frac{d^2 u}{d\vartheta^2} + u = \frac{P}{h^2 u^2} - \frac{T}{h^2 u^3} \frac{du}{d\vartheta},$$

$$\frac{d\tfrac{1}{2}h^2}{d\vartheta} = \frac{T}{u^3}, \quad \text{or} \quad \frac{d \log h}{d\vartheta} = \frac{T}{h^2 u^3};$$

so that
$$\frac{d^2 u}{d\vartheta^2} + u = \frac{P}{h^2 u^2} - \frac{d \log h}{d\vartheta} \frac{du}{d\vartheta},$$

and, since
$$P = -\frac{\partial U}{\partial r} = u^2 \frac{\partial U}{\partial u} = \mu u^2 + 2\nu u^3 \cos \vartheta,$$

$$T = \frac{\partial U}{r \partial \vartheta} = u \frac{\partial U}{\partial \vartheta} = \quad - \nu u^3 \sin \vartheta;$$

therefore our equations of motion are

$$\frac{d^2 u}{d\vartheta^2} = \frac{\mu}{h} + \frac{2\nu u}{h^2} \cos \vartheta + \frac{\nu}{h^2} \frac{du}{d\vartheta} \sin \vartheta,$$

$$\frac{d\tfrac{1}{2}h^2}{d\vartheta} = -\nu \sin \vartheta.$$

From these equations we find

$$\frac{d\tfrac{1}{2}v^2}{d\vartheta} = \frac{d}{d\vartheta}\tfrac{1}{2}h^2\left(\frac{du^2}{d\vartheta^2}+u^2\right)$$

$$= h^2\left(\frac{d^2u}{d\vartheta^2}+u\right)\frac{du}{d\vartheta}+\frac{d\tfrac{1}{2}h^2}{d\vartheta}\left(\frac{du^2}{d\vartheta^2}+u^2\right)$$

$$= \left(\mu+2\nu u\cos\vartheta+\nu\sin\vartheta\frac{du}{d\vartheta}\right)\frac{du}{d\vartheta}-\nu\sin\vartheta\left(\frac{du^2}{d\vartheta^2}+u^2\right)$$

$$= \mu\frac{du}{d\vartheta}+2\nu u\frac{du}{d\vartheta}\cos\vartheta-\nu u^2\sin\vartheta$$

$$= \frac{d}{d\vartheta}(\mu u+\nu u^2\cos\vartheta) = \frac{dU}{d\vartheta};$$

so that, integrating,

$$\tfrac{1}{2}h^2\left(\frac{du^2}{d\vartheta^2}+u^2\right) = U+H,$$

the equation of energy; and

$$\tfrac{1}{2}h^2 = C+\nu\cos\vartheta.$$

Therefore $$\frac{du^2}{d\vartheta^2}+u^2 = \frac{U+H}{C+\nu\cos\vartheta},$$

$$\frac{du^2}{d\vartheta^2} = \frac{H+\mu u-Cu^2}{C+\nu\cos\vartheta},$$

or $$\frac{du^2}{H+\mu u-Cu^2} = \frac{d\vartheta^2}{C+\nu\cos\vartheta},$$

a differential equation, in which the variables are separated.

Put $$\frac{2\nu}{C+\nu} = \kappa^2,\quad \frac{C-\nu}{C+\nu} = \kappa'^2,$$

then $$\frac{d\vartheta^2}{C+\nu\cos\vartheta} = \frac{1}{C+\nu}\,\frac{d\vartheta^2}{1-\kappa^2\sin^2\tfrac{1}{2}\vartheta};$$

so that $$\frac{(1-\tfrac{1}{2}\kappa^2)\,d\vartheta^2}{1-\kappa^2\sin\tfrac{1}{2}\vartheta} = \frac{C\,du^2}{H+\mu u-Cu^2}$$

$$= \frac{l^2\,du^2}{e^2-(lu-1)^2},$$

on putting $2C/\mu = l$, and $4CH/\mu^2 = e^2-1$.

Integrating, $$\cos^{-1}\frac{lu-1}{e} = \int\frac{\sqrt{(1-\tfrac{1}{2}\kappa^2)}\,d\vartheta}{\sqrt{(1-\kappa^2\sin^2\tfrac{1}{2}\vartheta)}}$$

$$= 2\sqrt{(1-\tfrac{1}{2}\kappa^2)}\,F(\tfrac{1}{2}\vartheta,\kappa),$$

or
$$\frac{l}{r} = 1 + e\cos\left\{2\sqrt{(1-\tfrac{1}{2}\kappa^2)}\,F(\tfrac{1}{2}\vartheta,\,\kappa)\right\},$$

the equation of the orbit, reducing, when $\nu = 0$ and $\kappa = 0$, to the ordinary elliptic orbit
$$\frac{l}{r} = 1 + e\cos\vartheta.$$

This would be the Earth's orbit, if we suppose the Sun's potential is partly magnetic, or if the Sun's strata, while still spherical, were in parts arranged slightly excentrically.

39. When the potential is entirely magnetic, $\mu = 0$; and
$$P = 2\nu u^3\cos\vartheta,\quad T = -\nu u^3\sin\vartheta.$$

Suppose the initial conditions such that $C = 0$; then
$$\tfrac{1}{2}h^2 = \nu\cos\vartheta,$$
so that $d\vartheta/dt = 0$ when $\vartheta = \pm\tfrac{1}{2}\pi$.

Now
$$\frac{d^2u}{d\vartheta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\vartheta} = u + \tfrac{1}{2}\tan\vartheta\frac{du}{d\vartheta},$$

or
$$\frac{d^2u}{d\vartheta^2}\bigg/\frac{du}{d\vartheta} = \tfrac{1}{2}\tan\vartheta;$$

and, integrating,
$$\frac{du^2}{d\vartheta^2} = A\sec\vartheta.$$

Then
$$\frac{dr^2}{dt^2} = \frac{dr^2}{d\vartheta^2}\frac{d\vartheta^2}{dt^2} = h^2u^4\frac{dr^2}{d\vartheta^2} = h^2\frac{du^2}{d\vartheta^2} = 2A\nu,$$

so that dr/dt is constant, and the particle approaches or recedes from the centre O at a constant rate.

The equation of the orbit is given by
$$au = \int\sqrt{(\sec\vartheta)}\,d\vartheta,$$
where
$$a = 1/\sqrt{A},$$
so that au is connected with ϑ by elliptic functions.

In the particular case where $A = 0$, we can put
$$u = \text{constant},\quad\text{or}\quad au = 1;$$
and now the particle oscillates on the semicircle from $\tfrac{1}{2}\pi$ to $-\tfrac{1}{2}\pi$, and
$$nt = \int\sqrt{(\sec\vartheta)}\,d\vartheta,$$
on putting
$$n^2 = 2\nu/a^4.$$

40. Generally in an orbit in which the radial velocity dr/dt is constant, and equal to V suppose, then

$$du/dt = -Vu^2,$$

while $$d\vartheta/dt = h,$$

so that $$h\,du/d\vartheta = -V;$$

and, differentiating logarithmically,

$$\frac{d\log h}{d\vartheta} = -\frac{d^2u}{d\vartheta^2}\bigg/\frac{du}{d\vartheta},$$

or $$\frac{d\log h}{d\vartheta}\,\frac{du}{d\vartheta} = -\frac{d^2u}{d\vartheta^2},$$

so that $P = h^2u^3$ in the orbit.

41. By placing SS' vertical, and supposing S' to recede vertically downwards to an infinite depth, while the strength B is proportionately increased, we obtain a field of force, partly due to gravity, and partly to an attracting centre at S.

For example, a particle placed on the horizontal line through S, at a distance such that gravity and the attraction of S are equal, will, when let go, oscillate on the arc of a parabola, with concavity upwards, from one end of the latus rectum to the other, the forces being at S.

Denoting this latus rectum by $4a$, the strength of the attracting centre at S will be $4ga^2$; and now, at distance r from the focus, the velocity will be

$$\sqrt{\left(\frac{8ga^2}{r} - 4ga - 2gr\right)},$$

the square root of the difference of the squares of the velocities in the free paths.

Such a state of motion might, perhaps, be carried out experimentally by drops of water liberated in the neighbourhood of an electrified sphere.

42. In the general investigation of the motion of a particle in this field of force, we must replace the relation

$$x + iy = c\cosh(\phi + i\vartheta)$$

by the new relation $$x + iy = \tfrac{1}{2}c\,(\phi + i\vartheta)^2;$$

and then $$x = \tfrac{1}{2}c\,(\phi^2 - \vartheta^2), \quad y = c\phi\vartheta;$$

and
$$r = \sqrt{(x^2+y^2)} = \tfrac{1}{2}c\,(\phi^2+\vartheta^2),$$
so that
$$r+x = c\phi^2, \quad r-x = c\vartheta^2,$$
representing confocal parabolas for constant values of ϕ and ϑ.

Now
$$T = \tfrac{1}{2}\,(\dot{x}^2+\dot{y}^2) = c^2\,(\phi^2+\vartheta^2)(\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}\dot{\vartheta}^2);$$
and, with a gravitation potential,
$$U = Ac^3r^{-1}+2Bcx = \frac{Ac^2}{\phi^2+\vartheta^2}+Bc^2\,(\phi^2-\vartheta^2).$$

Lagrange's equations of motion have two first integrals, namely,
$$T = U+Hc^2,$$
or
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)(\dot{\phi}+\dot{\vartheta}^2) = \frac{A}{\phi^2+\vartheta^2}+B\,(\phi^2-\vartheta^2)+H,$$
the equation of energy; and the second integral is
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)(\phi^2\dot{\vartheta}^2-\vartheta^2\dot{\phi}^2) = \tfrac{1}{2}A\,\frac{\phi^2-\vartheta^2}{\phi^2+\vartheta^2}-B\vartheta^2\phi^2+K = A\,\frac{x}{r}-B\,\frac{y^2}{c^2}+K.$$

Therefore
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)^2\,\dot{\phi}^2 = \tfrac{1}{2}A-K+H\phi^2+B\phi^4,$$
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)^2\,\dot{\vartheta}^2 = \tfrac{1}{2}A+K+H\vartheta^2-B\vartheta^4,$$
or
$$\frac{d\phi^2}{d\vartheta^2} = \frac{\tfrac{1}{2}A-K+H\phi^2+B\phi^4}{\tfrac{1}{2}A+K+H\vartheta^2-B\vartheta^4},$$
the differential equation of the orbit of the particle, in which the variables are separated; obtainable independently as a limiting case of the preceding results for confocal ellipses and hyperbolas, by replacing sin ϑ and sinh ϕ by ϑ and ϕ.

43. Suppose the parabola $\phi = \beta$, or a portion of it, is described by the particle; then
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)^2\,\dot{\phi}^2 = B\,(\phi^2-\beta^2)^2,$$
and the positive sign shows that the motion is unstable; also
$$\tfrac{1}{2}A-K = B\beta^4, \quad H = -2B\beta^2;$$
so that
$$\tfrac{1}{2}\,(\phi^2+\vartheta^2)^2\,\dot{\vartheta}^2 = A-B\,(\vartheta^2+\beta^2)^2.$$

If the particle oscillates on the arc of the parabola limited by $\vartheta = \alpha$ then
$$A/(\alpha^2+\beta^2)^2 = B,$$
and the forces are equal where the velocity is zero.

Now
$$\tfrac{1}{2}(\phi^2+\vartheta^2)^2\dot{\vartheta}^2 = B\left\{(\alpha^2+\beta^2)-(\vartheta^2+\beta^2)\right\}$$
$$= B(\alpha^2-\vartheta^2)(\alpha^2+2\beta^2+\vartheta^2);$$

so that
$$\frac{\pm d\phi}{\phi^2-\beta^2} = \frac{d\vartheta}{\sqrt{\{(\alpha^2-\vartheta^2)(\alpha^2+2\beta^2+\vartheta^2)\}}},$$

or
$$\frac{1}{\beta}\begin{matrix}\tanh^{-1}\\ \coth^{-1}\end{matrix}\frac{\phi}{\beta} = \frac{1}{\sqrt{(2\alpha^2+2\beta^2)}}\,\mathrm{cn}^{-1}\frac{\vartheta}{\alpha} + \text{constant},$$

with modulus $\alpha/\sqrt{(2\alpha^2+2\beta^2)}$, the equation of the disturbed orbit.

In the undisturbed orbit $\phi = \beta$,
$$\sqrt{(2B)}\frac{dt}{d\vartheta} = \frac{\beta^2+\vartheta^2}{\sqrt{\{(\alpha^2-\vartheta^2)(\alpha^2+2\beta^2+\vartheta^2)\}}},$$

so that t is given as a function of ϑ by elliptic integrals of the first and second kind.

44. When repelled from S, we change the sign of A, and write
$$\tfrac{1}{2}(\phi^2+\vartheta^2)^2\dot{\vartheta}^2 = -\tfrac{1}{2}A+K+H\vartheta^2-B\vartheta^4 = -B(\vartheta^2-\alpha^2)^2,$$

when the parabola $\vartheta = \alpha$ is described, the negative sign showing that this parabolic orbit is stable.

Then
$$\tfrac{1}{2}(\phi^2+\vartheta^2)^2\dot{\phi}^2 = B(\alpha^2+\phi^2)^2-A;$$

and if $\dot{\phi} = 0$ when $\phi = \beta$, so that $A = B(\alpha^2+\beta^2)^2$,

and the forces are equal, then
$$\tfrac{1}{2}(\phi^2+\vartheta^2)^2\dot{\phi}^2 = B(\phi^2-\beta^2)(2\alpha^2+\beta^2+\phi^2),$$

so that ϕ extends from β to infinity, and the unlimited part of the parabola $\vartheta = \alpha$ is described, the motion being stable; the time t, as before, being given by elliptic integrals of ϕ, of the first and second kind.

45. *Motion on Dipolar Circles.*

The polar reciprocals of these ellipses and hyperbolas, with respect to a focus S or S', are the non-intersecting dipolar circles, given by v = a constant in the equation
$$x+iy = c\tanh\tfrac{1}{2}(v+iu),$$

an equation which, when $u =$ a constant, represents circles passing through the poles S and S'; and these two systems of dipolar circles represent parallels and meridians on the stereographic projection, with S, S' as poles.

The Newtonian expression for the force to a point C, under which a circle APA' can be described, is

$$F = 2h^2 \,.\, AA'^2 \,.\, CP^{-2} \,.\, PV^{-3},$$

where PV is the chord of the circle through C. (*Principia*, Section II., Prop. VII., Part II.)

Professor Sylvester has shown ("Astronomical Prolusions," *Phil. Mag.*, Vol. XXXI., 1866), that this force can always be resolved into two forces, tending to S and S', and varying inversely as the fifth power of the distances from S and S'; while the circles passing through S and S', given by $u =$ constant, can be described under these forces, acting separately or conjointly.

Professor Sylvester also shows that this circular motion can take place under the attraction of a circular disc, bounded by one of the circles $u =$ constant, the particles of the disc attracting with intensity varying inversely as the fifth power of the distance.

46. With $$x + iy = c \tanh \tfrac{1}{2} (v + iu),$$

we find $$x = c \frac{\sinh v}{\cosh v + \cos u}, \quad y = c \frac{\sin u}{\cosh v + \cos u},$$

while $$T = \tfrac{1}{2} (\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2} c^2 (\cosh v + \cos u)^{-2} (\dot{u}^2 + \dot{v}^2).$$

Denoting, as before, SP and $S'P$ by r and s, we find

$$r^2 = \frac{2c^2 e^{-v}}{\cosh v + \cos u}, \quad s^2 = \frac{2c^2 e^{v}}{\cosh v + \cos u};$$

and supposing the strength of the attracting centres at S and S', varying inversely as the fifth powers of the distance, to be $4Ac^6$ and $4Bc^0$, the potential of their combined attraction

$$U = Ac^6 r^{-4} + Bc^6 s^{-4} = \tfrac{1}{4} c^2 (Ae^{2v} + Be^{-2v})(\cosh v + \cos u)^2;$$

so that the equation of energy, which we may write

$$T = U + Hc^4,$$

becomes

$$\tfrac{1}{2} (\cosh v + \cos u)^{-2} (\dot{u}^2 + \dot{v}^2) = \tfrac{1}{4} (Ae^{2v} + Be^{-2v})(\cosh v + \cos u)^2 + H.$$

47. We may form the Lagrangian equations of motion; but it is not possible now to obtain a second integral, in the form of Euler's second integral (7) for motion under two Newtonian centres of force, nor is it possible to obtain a differential equation of the orbit with the variables separated, in the general case.

For, if this were possible, we should obtain two equations of the form

$$\tfrac{1}{2}(\cosh v+\cos u)^{-4}\dot{u}^2=\tfrac{1}{2}n^2+Fu,$$

$$\tfrac{1}{2}(\cosh v+\cos u)^{-4}\dot{v}^2=\tfrac{1}{4}(Ae^{2v}+Be^{-2v})-\tfrac{1}{2}n^2+Gv;$$

and now tho equation of energy requires that

$$(\cosh v+\cos u)^2(Fu+Gv)$$

should be constant, a condition which cannot be satisfied.

48. By the application of the method of inversion, Mr. Larmor shows ("Least Action," *Proc. Lond. Math. Soc.*, Vol. xv., 1884, p. 162) that, with $H=0$, we can satisfy the equation by putting

$$\tfrac{1}{2}(\cosh v+\cos u)^{-4}\dot{u}^2=\tfrac{1}{2}n^2,$$

$$\tfrac{1}{2}(\cosh v+\cos u)^{-4}\dot{v}^2=\tfrac{1}{4}(Ae^{2v}+Be^{-2v})-\tfrac{1}{2}n^2,$$

so that
$$n\frac{dv^2}{du^2}=\tfrac{1}{2}(Ae^{2v}-2n^2+Be^{-2v}),$$

and the determination of the orbit is reduced to a quadrature, the variables being now separated.

49. In general, whenever we can put

$$x+iy=\mathrm{f}(u+iv),$$

so that u and v are conjugate functions of x and y, and whenever we can obtain the equation of energy in the form

$$\tfrac{1}{2}J^2(\dot{u}^2+\dot{v}^2)=Fu+Gv,$$

where J denotes the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$, and Fu, Gv are certain functions of u and v, then a possible state of motion is obtained by putting

$$\tfrac{1}{2}J^2\dot{u}^2=Fu+D,\quad \tfrac{1}{2}J^2\dot{v}^2=Gv-D;$$

so that
$$\frac{du^2}{dv^2}=\frac{Fu+D}{Gv-D},$$

the differential equation of the orbit, in which the variables are separated.

As a particular case, we can have Fu and Gv constant, and then du/dv is constant, and the orbit is an oblique trajectory of the curves u and v, equated to a constant.

50. When $$n^2 = \surd(AB),$$

$$\frac{dv^2}{du^2} = \tfrac{1}{2}\left(\surd A\, e^v - \surd B\, e^{-v}\right)^2,$$

so that we can have the circular orbit in which

$$v = v_0 = \tfrac{1}{4}\log B/A, \quad \text{or} \quad S'P/SP = \sqrt[4]{(B/A)},$$

the orbit discussed by Prof. Sylvester.

In this circular orbit, the resultant of the attractions $4Ac^6r^{-5}$ and $4Bc^6s^{-5}$, tending to S and S', will cut Ox in a point C, where

$$OC = c\,\frac{\sqrt[3]{B} - \sqrt[3]{A}}{\sqrt[3]{B} + \sqrt[3]{A}};$$

and Prof. Sylvester points out that the point P, moving in this circle APA' under a force to C, will follow a point Q moving like a planet in an ellipse on ACA' as major axis about a focus at C, so that PQ is always perpendicular to AA', and Prof. Sylvester then calls P the *excentric follower* of Q.

51. But the positive sign in the equation

$$\frac{dv^2}{du^2} = \tfrac{1}{2}\left(\surd A\, e^v - \surd B\, e^{-v}\right)^2$$

shows that this circular orbit is unstable, and that the orbit is ultimately given by

$$u = \int \frac{\pm\surd 2\, dv}{\surd A\, e^v - \surd B\, e^{-v}} = \sqrt{\left(\frac{2}{AB}\right)}\tanh^{-1}\frac{v}{v_0},$$

or $$\sqrt{\left(\frac{2}{AB}\right)}\coth^{-1}\frac{v}{v_0},$$

while $$nt = \int \frac{du}{(\cosh v + \cos u)^2};$$

so that in the unstable circular orbit, given by $v = v_0$, u and nt are connected together by the same relation as that between *true* and *mean anomaly* in an elliptic orbit, of excentricity $\operatorname{sech} v_0$.

With A and B of opposite sign, this circular orbit would be stable.

52. In the circular orbits passing through S and S', we take u = constant, $\dot{u} = 0$, and then $H = 0$, and

$$\dot{v}^2 = \tfrac{1}{2}(Ae^{2v}+Be^{-2v})(\cosh v+\cos u)^4,$$

$$t = \int \frac{\sqrt{2}\,dv}{(\cosh v+\cos u)^2\sqrt{(Ae^{2v}+Be^{-2v})}},$$

giving t as a function of v, and reducing to well known results, when

$$A = 0, \quad \text{or} \quad B = 0.$$

53. As in the confocal elliptic and hyperbolic orbits, we can also have circles described under appropriate forces, with

$$\vartheta = nt, \quad \text{or} \quad \phi = nt.$$

May 14*th*, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. H. R. Norris, B.A. Camb., M.A., B.Sc. London, late Scholar of St. John's College, Cambridge, was elected a member; and Mr. G. H. Bryan was admitted into the Society. Prof. G. W. Jones, of Cornell University, Ithaca, New York, was present as a visitor.

The following communications were made:—

- Relations between the Divisors of the First n Numbers: Dr. Glaisher.
- Wave Motion in a Heterogeneous Heavy Liquid: Mr. A. E. H. Love.
- Disturbance produced by an Element of a Plane Wave of Sound or Light: Mr. A. B. Basset.
- On Functions determined from their Discontinuities and a certain Form of Boundary Condition: Prof. W. Burnside.
- On a certain Riemann's Surface: Prof. W. Burnside.

Messrs. Macmahon, Larmor, Bryan, and the President took part in discussions on the papers.

The following presents were received :—

"The Great Trigonometrical Survey of India," Vols. xi., xii., xiii., 4to; Dehra Dun, 1890.

"Proceedings of the Royal Society," Vol. xlix., No. 298.

"Journal of the Institute of Actuaries," Vol. xxix., Part ii., April, 1891.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. iv., No. 3, 1890–1.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. liv.

"Educational Times," for May.

"Annals of Mathematics," Vol. v., No. 6; Feb., 1891, Virginia.

"Bulletin des Sciences Mathématiques," Index to Vol. xiv., Tome xv., Mars, 1891.

"Bulletin de la Société Mathématique de France," Tome xix., No. 3; Paris, 1891.

"Rendiconti del Circolo Matematico di Palermo," Tomo v., Fasc. iii.; Estratto —Sophie de Kowalevski.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stück 4.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 128, and one No. of Index.

"Memorias y Revista de la Sociedad Cientifica 'Antonio Alzate,'" Tomo iv., Nos. 5 and 6.

"Nyt Tidsskrift for Mathematik," A, Anden Aargang, Nr. 1 and 2; B, Anden Aargang, Nr. 1; Kjobenhavn, 1891.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxv., 1re Livraison; Harlem, 1891.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vii., Fasc. vii.; Roma, 1891.

"Annali di Matematica," Tomo xix., Fasc. 2; Milano, 1891.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. iv., Fasc. 1–12.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg," Band i., No. 9; Band iii., Heft 1.

"Abhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften," Band xvii., No. 2.

"Journal für die reine und angewandte Mathematik," Band cviii., Heft 1.

Cabinet Likenesses of Mr. Ernest Carpmael, and Dr. M. J. M. Hill, for the Album.

Wave-Motion in a Heterogeneous Heavy Liquid.

By Mr. A. E. H. Love.

[*Read May 14th*, 1891.]

In *Proceedings*, Vol. xx., pp. 392 *seq.*, Mr. Burnside has discussed the theory of wave-motion in a heterogeneous fluid. The density is supposed to be some function of the depth in the equilibrium configuration, and the heterogeneous fluid is treated as the limit of an infinite number of layers of homogeneous fluid of different densities. The equations of motion of each layer are formed, and the conditions that hold at an interface, and the equations applicable to the problem in hand are obtained by subtracting like equations relative to consecutive layers. This method is open to the objection that, as the layers are afterwards supposed indefinitely thin, the elevation of the waves may be of any order of magnitude in comparison with the thickness of the layers. Although this objection might be removed, it appeared to me to be worth while to form directly the equations of the stream-function for wave propagation in a heterogeneous heavy fluid. Mr. Burnside considers in greatest detail the case where the density increases in geometric progression as the depth increases in arithmetic progression. I have obtained the solution of the equation of the stream-function in this case, thus finding the nature of the motion, and the relation between wave-length and velocity of propagation. This relation has been given by Mr. Burnside. I also consider the case of infinite depth, with special reference to the horizontal propagation of gravitational waves in an atmosphere resting on a horizontal plane and otherwise unlimited.

1. Suppose a heterogeneous incompressible fluid under the action of gravity. Take the axis of y vertically upwards from the free surface, and suppose that when undisturbed the density ρ_0 is given by the equations

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y} = -\kappa', \quad \rho_0 = Ce^{-\kappa' y} \qquad\qquad (1),$$

and that ρ_0 is independent of x.

When the fluid is slightly disturbed, the equations of small motion will be

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial v}{\partial t} &= -g-\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\}\tag{2}$$

Since the fluid is incompressible, we must have

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}=0 \tag{3}$$

and the equation of continuity becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

In virtue of (4), u and v are given by a stream-function ψ, such that

$$d\psi = -v\,dx+u\,dy,$$

and thus the equations of motion and incompressibility become

$$\left.\begin{aligned}\frac{\partial^2\psi}{\partial y\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial^2\psi}{\partial x\partial t} &= g+\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\}\tag{5}$$

and

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho\psi)}{\partial(xy)}=0 \tag{6}$$

Eliminating p from (5), we obtain

$$\rho\frac{\partial}{\partial t}(\nabla^2\psi)-g\frac{\partial \rho}{\partial x}+\frac{\partial \rho}{\partial x}\frac{\partial^2\psi}{\partial x\partial t}+\frac{\partial \rho}{\partial y}\frac{\partial^2\psi}{\partial y\partial t}=0 \tag{7}$$

Now $\frac{\partial \rho}{\partial x}$ is a small quantity of the order of the amplitude of the waves, and thus (6) and (7) may be written

$$\frac{\partial \rho}{\partial t}-\frac{\partial \rho}{\partial y}\frac{\partial \psi}{\partial x}=0 \tag{8}$$

and

$$\rho\frac{\partial}{\partial t}(\nabla^2\psi)-g\frac{\partial \rho}{\partial x}+\frac{\partial \rho}{\partial y}\frac{\partial^2\psi}{\partial y\partial t}=0 \tag{9}$$

In the small terms of (8) and (9), we may put for ρ and $\frac{\partial \rho}{\partial y}$ their values from (1), and thus get

$$\frac{\partial\rho}{\partial t}+\kappa'\rho_0\frac{\partial\psi}{\partial x}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

and

$$\rho_0\frac{\partial}{\partial t}(\nabla^2\psi)-g\frac{\partial\rho}{\partial x}-\kappa'\rho_0\frac{\partial^2\psi}{\partial y\,\partial t}=0\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Differentiating (11) with respect to t, and eliminating $\frac{\partial^2\rho}{\partial x\,\partial t}$, by using (10), we find an equation which can be divided through by ρ_0, giving the result

$$\frac{\partial^2}{\partial t^2}(\nabla^2\psi)+g\kappa'\frac{\partial^2\psi}{\partial x^2}-\kappa'\frac{\partial^3\psi}{\partial y\,\partial t^2}=0\;\ldots\ldots\ldots\ldots\ldots(12),$$

which is the differential equation for the stream-function ψ.

Now suppose the waves are of length λ, and propagated with velocity V, and write

$$m=2\pi/\lambda,\quad n=2\pi V/\lambda,$$

and assume, to satisfy (12),

$$\psi=F(y)\,e^{\iota(mx-nt)}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Then $F(y)$ will have to satisfy the equation

$$\frac{d^2F}{dy^2}-\kappa'\frac{dF}{dy}+m^2\left(\frac{g\kappa'}{n^2}-1\right)F=0\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

And the solution may be written

$$F=Ae^{\frac{1}{2}\kappa' y}\sin\mu\,(y+\delta)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

in which δ is the whole depth, and

$$\mu^2=m^2\left(\frac{g\kappa'}{n^2}-1\right)-\frac{\kappa'^2}{4}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

2. We have next to find the pressure. Assume that

$$\rho=\rho_0+\phi\,(y)\,e^{\iota(mx-nt)},$$

then by (10) we find

$$-\phi\,(y)\,\iota n+\kappa'\rho_0\iota mF(y)=0,$$

so that

$$\rho=\rho_0\left(1+\kappa'\frac{m}{n}\psi\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Now we have, by (5),

$$\left.\begin{aligned}\frac{\partial p}{\partial x}&=-\rho_0\left(1+\kappa'\frac{m}{n}\,\psi\right)\frac{\partial^2\psi}{\partial y\,\partial t}\\ \frac{\partial p}{\partial y}&=-g\rho_0\left(1+\kappa'\frac{m}{n}\,\psi\right)+\rho_0\left(1+\kappa'\frac{m}{n}\,\psi\right)\frac{\partial^2\psi}{\partial x\,\partial t}\end{aligned}\right\},$$

or, neglecting products and putting in the values of ρ_0, ψ, these are

$$\left.\begin{aligned}\frac{\partial p}{\partial x} &= Ce^{-\kappa' y}\iota n\frac{\partial F}{\partial y}e^{\iota(mx-nt)}\\ \frac{\partial p}{\partial y} &= -gCe^{-\kappa' y}\left[1+\frac{\kappa' m}{n}F(y)\,e^{\iota(mx-nt)}\right]+Ce^{-\kappa' y}mn\,F(y)\,e^{\iota(mx-nt)}\end{aligned}\right\}\ldots(18).$$

Equation (14) secures the satisfaction of the condition of integrability, and therefore we have only to integrate an expression of the form $Xdx+Ydy$, known to be a perfect differential. Integrating the first equation partially with respect to x, and adding the part that arises from the first term of the second equation, we have

$$p = \text{const.}+\frac{g}{\kappa'}Ce^{-\kappa' y}$$

$$+C\frac{n}{m}Ae^{-\frac{1}{2}\kappa' y}\left[\tfrac{1}{2}\kappa'\sin\mu\,(y+\delta)+\mu\cos\mu\,(y+\delta)\right]e^{\iota(mx-nt)}\ldots\ldots(19).$$

3. Suppose the equation of the disturbed free surface to be

$$y-be^{\iota(mx-nt)} = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

where the modulus of b is small of the same order as A. Then

$$p = 0$$

when

$$y = be^{\iota(mx-nt)},$$

and substituting in (19) we find that

$$\frac{g}{\kappa'}e^{-\kappa' be^{\iota(mx-nt)}}+\frac{n}{m}Ae^{\iota(mx-nt)}\left(\tfrac{1}{2}\kappa'\sin\mu\delta+\mu\cos\mu\delta\right)$$

must be constant. Expanding the exponential in the first term, and rejecting powers of b above the first, we have

$$-gb+\frac{n}{m}A\left(\tfrac{1}{2}\kappa'\sin\mu\delta+\mu\cos\mu\delta\right) = 0\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Again, since equation (20) is the equation of a surface which always contains the same particles, we must have

$$b\iota n\,e^{\iota(mx-nt)}-\frac{\partial\psi}{\partial x} = 0$$

at this surface, so that we have a second relation between A and b, *viz.*:—

$$bn-mA\sin\mu\delta = 0.$$

Eliminating A and b, we get

$$g \sin \mu\delta - \frac{n^2}{m^2}(\mu \cos \mu\delta + \tfrac{1}{2}\kappa' \sin \mu\delta) = 0,$$

or

$$\tan \mu\delta = \mu \frac{V^2}{g - \frac{1}{2}\kappa' V^2} \qquad \text{.......................(24).}$$

This equation can be identified with Mr. Burnside's in *Proceedings*, Vol. xx., p. 396, by writing m' for κ', $\frac{1}{2}i(p_2 - p_1)$ for μ, and H for δ.

If μ be imaginary we may write

$$\psi = Ae^{\frac{1}{2}\kappa' y} \sinh m'(y+\delta)\, e^{i(mx-nt)} \qquad \text{.................... (25),}$$

where

$$m'^2 = m^2 + \tfrac{1}{4}\kappa'^2 - \frac{g\kappa'}{V^2} = -\mu^2 \qquad \text{.................... (26),}$$

and then equation (24) becomes

$$\tanh m'\delta = m' \frac{V^2}{g - \frac{1}{2}\kappa' V^2} \qquad \text{.....................(27).}$$

4. In the discussion of these equations, we shall suppose the velocity to be given, and seek to determine the wave-length, and we shall first suppose that $g - \frac{1}{2}\kappa' V^2$ is positive and $> V^2/\delta$.

Now, if $$1 < \delta(g/V^2 - \tfrac{1}{2}\kappa) < \infty,$$

the equation (27) has one real root, and there is one real wave-motion of the type (25). The relation between velocity and wave-length is

$$\frac{4\pi^2}{\lambda^2} = \frac{g\kappa'}{V^2} - \tfrac{1}{4}\kappa'^2 + \frac{\alpha^2}{\delta^2} \qquad \text{........................ (28),}$$

where α is the real root of the equation

$$\tanh x = \frac{x}{\delta(g/V^2 - \frac{1}{2}\kappa')} \qquad \text{........................ (29),}$$

and the right-hand side of the equation for λ is positive, since

$$g\kappa'/V^2 > \tfrac{1}{2}\kappa'^2.$$

In the same case of

$$1 < \delta(g/V^2 - \tfrac{1}{2}\kappa') < \infty,$$

the equation (24) has an infinite number of roots, of which the first is greater than π. Let $\alpha_1, \alpha_2, \ldots$ be these roots, then the wave-length for a given velocity is given by the equation

$$\frac{4\pi^2}{\lambda^2} = \frac{g\kappa'}{V^2} - \tfrac{1}{4}\kappa'^2 - \frac{\alpha_r^2}{\delta^2}, \quad (r = 1, 2, \ldots) \qquad \text{.............. (30),}$$

and there will be a corresponding real wave-motion if the right-hand side be positive. Now the α_r are roots of the equation

$$\tan x = \frac{x}{\delta\,(g/V^2 - \frac{1}{2}\kappa')} \quad\text{.........................(31)},$$

where the denominator of the right-hand side is positive and greater than unity, and we know that the higher roots of this equation approximate to odd multiples of $\frac{1}{2}\pi$, and that the lowest root is greater than π. Thus there will be in general, for a sufficiently small velocity V, a certain number of modes of motion of the type expressed by equations (13) and (15). Supposing r the number of modes, then α_r is the greatest root of (31) for which the right-hand side of (30) is real, and α_r is approximately $\frac{1}{2}(2r+1)\pi$. In the case under consideration the stream-function contains the factor $\sin\mu(y+\delta)$, which is $\sin\frac{2r+1}{2\delta}\pi(y+\delta)$ approximately, and this vanishes when

$$y/\delta = -1 + 2N/(2r+1),$$

where N is an integer. Since $\delta > -y > 0$, we may take $N = 1, 2, \dots r$, and thus there are r horizontal equidistant planes at which ψ vanishes. The vertical component of the velocity vanishes at these planes, so that we may state our results in the form:—

For given velocity of propagation there will be a definite number $(r+1)$ of possible modes of motion. Of these one is defined by an equation like (25), and there is no nodal plane except the fixed bottom. The others are defined by equations like (13) and (15), and they have respectively 1, 2, ... r nodal planes.

In the first mode, for which there is no nodal plane, the motion, so far as it depends on the depth, is defined by exponential functions. The increment of pressure due to the wave-motion is proportional to the product of the original density and the horizontal velocity, and the increment of density is proportional to the product of the original density and the stream-function.

In the remaining r modes, which have respectively 1, 2, ... r nodal planes, the motion, so far as it depends upon depth, may be regarded as defined by a simple harmonic function of the depth with an amplitude diminishing in geometric progression as the depth increases in arithmetic progression. In these modes of motion the fluid is divided into strata all but the highest of which are of equal thickness, and across the bounding planes of which there is no flow, and the kinetic energy is the same in all the strata. The increments of pressure and density are connected with the velocity components in the same manner as in the other mode.

When $g/V^2-\frac{1}{2}\kappa'$ is positive, and

$$0<\delta\,(g/V^2-\tfrac{1}{2}\kappa')<1,$$

the equation (29) has no real root; the first root of (31) is $<\frac{1}{2}\pi$, and the first of the corresponding modes has no nodal plane.

When $g/V^2-\frac{1}{2}\kappa'$ is negative, equation (29) has no real root, and the first of the roots of (31) lies between $\frac{1}{2}\pi$ and π, and in this case also the corresponding mode has no nodal plane.

In these two cases all the modes are similarly expressed to the r nodal modes in the previous case.

All the modes which we have been discussing are rotational. They include every possible gravitational oscillation that can be propagated horizontally in liquid of finite depth with the given law of density.

5. Our solution may be applied to the case of infinite depth. Supposing in the first place that μ is imaginary, we take for ψ the form

$$\psi=e^{\frac{1}{2}\kappa' y}\left(Ae^{m'y}+Be^{-m'y}\right)e^{\iota(mx-nt)}.$$

Then, if $\frac{1}{2}\kappa'>m'$, we must retain both A and B, but if $\frac{1}{2}\kappa'<m'$, we must reject B.

(i.) When $\frac{1}{2}\kappa'>m'$, we may write the above

$$\psi=Ae^{\frac{1}{2}\kappa' y}\sinh m'(y+h)\,e^{\iota(mx-nt)},$$

then the motion above the plane $y=-h$ is the same as the motion with imaginary μ in a fluid of the same law of density, and of finite depth h.

We have the four relations

$$m'^2=\tfrac{1}{4}\kappa'^2+\frac{4\pi^2}{\lambda^2}-\frac{g\kappa'}{V^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(26),$$

$$\tanh m'h=\frac{m'h}{h\,(g/V^2-\frac{1}{2}\kappa')}\ \ldots\ldots\ldots\ldots\ldots\ldots(29),$$

$$1<h\,(g/V^2-\tfrac{1}{2}\kappa')<\infty\ \ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

$$m'<\tfrac{1}{2}\kappa'\ \ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

Now suppose m' chosen so as to satisfy the inequality (β) and

$h\,(g/V^2-\frac{1}{2}\kappa')=\dfrac{1}{a}$, say, so as to satisfy the inequality (α). Then, by (29), h is determined in terms of m' and a, and therefore V in terms of m' and a, and then λ can be found from (26), so that there is one definite mode of motion associated with the assumed data. This mode is rotational and has one nodal plane at $y=-h$.

(ii.) When $\frac{1}{2}\kappa'<m'$, we have to take for ψ the form

$$\psi = Ae^{\frac{1}{2}\kappa' y+m'y+\iota(mx-nt)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32),$$

and the pressure equation becomes

$$p = Ce^{-\kappa' y}\left[\frac{g}{\kappa'}+\frac{n}{m}\,(\tfrac{1}{2}\kappa'+m')\,Ae^{(\frac{1}{2}\kappa'+m')\,y+\iota(mx-nt)}\right]\ \ldots\ldots(33).$$

Supposing, as before, that the equation of the free surface is

$$y-b\,e^{\iota(mx-nt)}=0,$$

we make this a surface of constant pressure, by putting

$$-gb+\frac{n}{m}\,(\tfrac{1}{2}\kappa'+m')\,A=0,$$

and we make it always contain the same particles by putting

$$b\iota n-A\iota m=0,$$

whence, eliminating A and b, we get

$$m'=g/V^2-\tfrac{1}{2}\kappa'\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

which satisfies the inequality $\frac{1}{2}\kappa'<m'$, if $g/V^2>\kappa'$.

The wave-length is in this case given by the equation

$$\frac{4\pi^2}{\lambda^2}=\left(\frac{g}{V^2}-\tfrac{1}{2}\kappa'\right)^2+\frac{g\kappa'}{V^2}-\tfrac{1}{4}\kappa'^2=\frac{g^2}{V^4},$$

so that
$$V^2=\tfrac{1}{2}g\lambda/\pi,$$

and the motion is irrotational, and there is no nodal plane.

We may also apply the solutions in which μ is real. It is best to take

$$\mu\delta = 2N\pi+\mu h,$$

where N is infinite and h finite but undetermined, then h is the depth of a nodal plane. The stream-function is

$$Ae^{\frac{1}{2}\kappa' y}\sin\mu\,(y+h)\,e^{\iota(mx-nt)},$$

and the relation between velocity and wave-length is

$$\frac{g\kappa'}{V^2}=\frac{4\pi^2}{\lambda^2}+\tfrac{1}{4}\kappa'^2+\mu^2,$$

where μ is connected with h by the equation

$$\tan\mu h=\frac{\mu}{g/V^2-\frac{1}{2}\kappa'}.$$

Thus the motion above the nodal plane $y=-h$ takes place precisely as if this plane were a fixed bottom.

The conclusion is that, in addition to the irrotational mode, the liquid of infinite depth with the given law of density possesses all the modes of oscillation of a liquid of any finite depth whatever and the same law of density.

Mathematically considered, the solutions for infinite depth are in a sense simpler than those for finite depth, inasmuch as there is no relation between velocity and wave-length. Within certain limits these may be any whatever. The motion may be regarded as partly defined by the velocity (or the wave-length), and partly by the quantity μ (or m'), and then there is an *algebraic* relation between the velocity, the wave-length, and μ (or m'). If μ (or m') be associated with the depth of a nodal plane, we obtain an interpretation which makes the matter clearer physically, at the expense of introducing a transcendental equation into the mathematics. In the case of finite depth there is always a relation between velocity and wave-length, depending on a *transcendental* equation.

6. We may also consider the case of fluid unlimited upwards. The problem is that of wave-motion in a heterogeneous fluid resting on an infinite horizontal plane, and the law of density is the same as that of the atmosphere near the earth's surface; so that our theory will have an application to gravitational waves in the atmosphere propagated horizontally, provided the velocity of propagation is small in comparison with the velocity of sound. There can be little doubt that viscosity and other causes will hold in check the tendency to

indefinite increase in the motion at considerable heights, so that our solution may be regarded as approximately true near the earth's surface.

The solutions when μ is imaginary will be in this connexion unimportant, and we may pass to the consideration of cases with μ real, for which there are nodal planes. The stream-function is of the form

$$Ae^{\frac{1}{2}\kappa' y}\sin\mu y\, e^{\iota(mx-nt)}.$$

Now we may assume any distance h we please as the distance of the nearest nodal plane from the horizontal fixed bottom, and then π/h is μ, and the equation connecting velocity and wave-length is

$$\frac{g\kappa'}{V^2}=\frac{\pi^2}{h^2}+\frac{4\pi^2}{\lambda^2}+\tfrac{1}{4}\kappa'^2.$$

If κ be the constant ratio of pressure to density, $\kappa'=g/\kappa$, and the equation becomes

$$\frac{g^2}{\kappa V^2}=\frac{\pi^2}{h^2}+\frac{4\pi^2}{\lambda^2}+\tfrac{1}{4}\frac{g^2}{\kappa^2}.$$

If this has anything to do with atmospheric conditions, it would probably be not unreasonable to suppose h comparable with a mile, and λ large compared with h. Now, remembering that $\sqrt{\kappa}$ is the Newtonian velocity of sound ($=28\times10^3$ centimetres per second, approximately), we see that the last term is negligible, and, neglecting the second term, the equation reduces to

$$V=\frac{gh}{\pi\sqrt{\kappa}}\text{ per sec.},$$

and this is about $40h$ per hour. Thus, if the height of the nearest nodal plane be half-a-mile, the velocity of propagation of the disturbance is twenty miles an hour.

On the Disturbance produced by an Element of a Plane Wave of Sound or Light. By A. B. BASSET.

[*Read May* 14*th*, 1891.]

1. The effect of the disturbance produced at a distant point by an element of a plane wave of light, was first investigated by Sir G. Stokes,* who showed that, upon the assumption that the equations of motion of the luminiferous ether are of the same form as those of an isotropic elastic solid, the disturbance produced at a distant point P by an element dS of the plane wave

$$\zeta = c \sin \frac{2\pi}{\lambda}(bt-x),$$

is given by the equation

$$\zeta' = \frac{c\,dS}{2\lambda r}(1+\cos\theta)\sin\phi\cos\frac{2\pi}{\lambda}(bt-r) \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

where r is the distance of the element dS from P, and θ and ϕ are the angles which the direction of r makes with the direction of propagation and vibration respectively.

The method employed by Stokes is equally applicable to the electromagnetic theory of light, inasmuch as the equations of electric displacement (upon Maxwell's theory), are of the same form as those which are satisfied by those portions of the displacement of an elastic solid upon which distortion unaccompanied by dilatation depends.

The question of the resolution of plane waves has recently been discussed by Lord Rayleigh,† who has suggested that the method of resolution employed by Stokes is the simplest, but only one of an indefinite number which might be proposed, all of which are equally legitimate so long as the question is regarded as a purely mathematical one; and that the purely mathematical question has no definite answer. I propose, in the present paper, to consider the question raised by Lord Rayleigh in detail; but it will be convenient to commence with the case of waves of sound, because the analysis, depending as it does on a single function, is far more simple than in the case of light.

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* "Dynamical Theory of Diffraction," *Trans. Camb. Phil. Soc.*, Vol. IX., p. 1; and *Math. and Phys. Papers*, Vol. II., p. 243.

† *Encycl. Brit.*, Art. "Wave Theory," pp. 452-454.

2. To fix our ideas, let us suppose that a source of sound acts for a very short time and then ceases. A spherical wave will be generated, which will be propagated through space, and the disturbance at any particular epoch will be confined to a thin sheet bounded by two spheres concentric with the source. Any point P outside this space will remain at rest until the outer boundary of the sheet reaches it; it will then be in a state of disturbance until the inner boundary reaches it, and, after the latter has passed over it, P will remain at rest for ever. If the source be periodic, the disturbance produced by it may be conceived to consist of a succession of very thin sheets, each one of which as it passes over P creates a disturbance, and then travels on, leaving no trace of its existence behind it.

Let us now take up our position at a point upon a *fixed* sphere whose centre is the source, and let t be the time which the wave occupies in travelling from this sphere to an external point P. Then the disturbance which actually exists at P at any particular instant, must depend upon that which existed at the surface of this sphere at a previous epoch t. And since every element of the sphere may be supposed to contribute something to the disturbance at P, it follows that the disturbance produced by an element dS must be equal to σdS, where σ is to be determined from the condition, that the value at P of the integral $\iint \sigma dS$ taken over the whole surface of the sphere, is equal to the disturbance which actually exists at P, whilst its value at any internal point is zero.

The function σ must be capable of being represented mathematically by a series of multiple sources of sound; and it therefore follows that, in order to determine the effect at P of the successive waves which are continually being transmitted across the sphere, we may, if we please, suppose the original source of sound to be removed, and a spherical sheet of multiple sources to be substituted for it, which at an external point produces the same effect as the original source, and whose effect at an internal point is zero.

The question we have to consider is, whether this distribution of multiple sources is unique, or whether there are an infinite number of distributions which will produce the required effect; and we shall show that the latter is the case.

3. The simplest case which we can have, is that of plane waves of sound which are travelling parallel to the axis of x, and whose velocity potential (the time factor $\epsilon^{\iota\kappa a t}$ being omitted) may be taken to be

$$\varphi = \epsilon^{-\iota\kappa x};$$

and we have to consider what distribution of sources over the plane $x = 0$ will produce this velocity potential when x is positive, and zero velocity potential when x is negative.

It is evident that the required distribution cannot consist of *simple* sources, since a simple source would produce a disturbance on the negative side of the plane; if, however, we combine a simple source with a double source whose axis is perpendicular to the plane, the required effect may be produced, and we shall now show that this will be the case if we take

$$\sigma = \frac{\iota\kappa\epsilon^{-\iota\kappa r}}{4\pi r} + \frac{\iota\kappa x}{4\pi r^2}\left(1 + \frac{1}{\iota\kappa r}\right)\epsilon^{-\iota\kappa r} \quad \text{.....................} \quad (2),$$

where $\kappa = 2\pi/\lambda.$

Restoring the time factor, the real part of this expression is

$$\sigma = -\frac{1}{2\lambda r}(1+\cos\theta)\sin\frac{2\pi}{\lambda}(at-r) + \frac{\cos\theta}{4\pi r^2}\cos\frac{2\pi}{\lambda}(at-r) \quad \text{......} \quad (3),$$

where θ is the angle which the line joining the element to P, makes with the normal to the wave front drawn in the direction of propagation.

Whence, if x is positive, $\cos\theta = x/r$, and

$$\iint \sigma dS = -\frac{\pi}{\lambda}\int_x^\infty \left(1 + \frac{x}{r}\right)\sin\frac{2\pi}{\lambda}(at-r)\,dr + \tfrac{1}{2}x\int_x^\infty \frac{1}{r^2}\cos\frac{2\pi}{\lambda}(at-r)\,dr$$

$$= \cos\frac{2\pi}{\lambda}(at-x).$$

If, however, x is negative, we must put $\cos\theta = -x/r$, and the integral will be zero.

4. We have now to consider whether this distribution is the only one which can produce the required effect.

The most general expression for a multiple divergent source which is symmetrical with respect to the axis of x is*

$$\Psi_n = \psi_n P_n,$$

where P_n is a zonal harmonic, and

$$\psi_n = \frac{\epsilon^{-\iota\kappa r}}{r} f_n(\iota\kappa r),$$

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* Stokes, *Phil. Trans.*, 1868; and Lord Rayleigh, *Theory of Sound*, Chap. XVII.

where

$$f_n(x) = 1 + \frac{n(n+1)}{2x} + \frac{(n-1)n(n+1)(n+2)}{2 \,.\, 4x^2} + \dots$$

$$\dots + \frac{1 \,.\, 2 \,.\, 3 \dots 2n}{2 \,.\, 4 \,.\, 6 \dots 2n\, x^n} \dots\dots\dots\dots \text{(4).}$$

The function ψ can also be shown to satisfy the following equations, viz.,

$$r\frac{d\psi_n}{dr} - n\psi_n = -\iota\kappa r\psi_{n+1} \dots\dots\dots\dots \text{(5)},$$

$$r\frac{d\psi_n}{dr} + (n+1)\psi_n = -\iota\kappa r\psi_{n-1} \dots\dots\dots\dots \text{(6)},$$

$$(2n+1)\psi_n = \iota\kappa r(\psi_{n+1} - \psi_{n-1}) \dots\dots\dots\dots \text{(7)},$$

which are frequently useful.

Now, if ϕ and ψ be any functions which satisfy the equation

$$(\nabla^2 + \kappa^2)\phi = 0,$$

and which vanish at infinity, it follows that, if we apply Green's theorem to any periphractic region,

$$\iint \phi \frac{d\psi}{d\nu} dS = \iint \psi \frac{d\phi}{d\nu} dS \dots\dots\dots\dots \text{(8)},$$

where the surface integrals are to be taken over the internal boundaries. If, however, ϕ or ψ become infinite at any point of the region, we must describe a small sphere round such a point as centre, take the surface integrals over the surface of this sphere, and then make the radius of the latter zero.

Let

$$\phi = \epsilon^{-\iota\kappa x},$$

$$\psi = A\psi_n P_n;$$

where x is the distance of any point P on the axis of x from the plane of resolution (that is, the plane $x = 0$), r is the distance of any point on this plane from P, and θ is the angle which the direction of r makes with the normal and the plane drawn in the direction of propagation.

Applying Green's theorem to the region lying on the positive side of the plane $x = 0$, it follows that, if P lies on the negative side of this plane, ψ does not become infinite at any point of the region considered; whence (8) becomes

$$\iint \frac{d\psi}{d\nu} dS + \iota\kappa \iint \psi\, dS = 0 \dots\dots\dots\dots \text{(9).}$$

If, however, P lies on the positive side of the plane, ψ becomes infinite at P, whence (8) becomes

$$\iint \frac{d\psi}{d\nu} dS + \iota\kappa \iint \psi \, dS = -c^2 \epsilon^{-\iota\kappa x} \iint \epsilon^{\iota\kappa c \cos\theta} \frac{d\psi}{dc} \, d\Omega$$

$$+ c^2 \epsilon^{-\iota\kappa x} \iint \psi \frac{d}{dc} (\epsilon^{\iota\kappa c \cos\theta}) \, d\Omega \quad \text{.................(10)};$$

where dS is an element of the plane $x = 0$, and $d\Omega$ is the solid angle subtended at P by an element of the surface of a sphere of radius c described about P as centre.

Now* if
$$\mu = \cos\theta,$$
$$\epsilon^{\iota\kappa c \cos\theta} = \Sigma_0^\infty F_i(c) P_i(\mu),$$
where
$$F_i(c) = \frac{(\iota\kappa c)^i}{1.3.5\ldots(2i-1)} \left\{ 1 - \frac{\kappa^2 c^2}{2.2i+3} + \frac{\kappa^4 c^4}{2.4.2i+3.2i+5} - \ldots \right\},$$
whence the right-hand side of (10) becomes

$$A c^2 \epsilon^{-\iota\kappa x} \iint \left\{ \psi_n P_n \Sigma \left(\frac{dF_i}{dc} P_i \right) - \frac{d\psi_n}{dc} P_n \Sigma (F_i P_i) \right\} d\Omega$$

$$= \frac{4\pi A c^2 \epsilon^{-\iota\kappa x}}{2n+1} \left(\psi_n \frac{dF_n}{dc} - F_n \frac{d\psi_n}{dc} \right).$$

Substituting the values of ψ_n and F_n, and making $c = 0$, it will be found that the right-hand side is equal to $4\pi A \epsilon^{-\iota\kappa x}$; and, since this is to be equal to $\epsilon^{-\iota\kappa x}$, we must have $4\pi A = 1$, whence on the positive side of the plane

$$\iint \frac{d\psi}{d\nu} dS + \iota\kappa \iint \psi dS = \epsilon^{-\iota\kappa x} \quad \text{................. (11)},$$

where
$$\psi = \frac{\psi_n P_n}{4\pi} \quad \text{.............................. (12)}.$$

Since $d/d\nu = -d/dx$, it follows that the effect of an element dS is represented by the expression

$$\left(\iota\kappa\psi - \frac{d\psi}{dx} \right) dS \quad \text{..............................(13)},$$

where ψ is given by (12). The first term of this expression represents a multiple source of the n^{th} order; to find what the second term is, we have

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* Rayleigh, *Theory of Sound*, § 334.

$$\frac{d}{dx}=\mu\frac{d}{dr}+\frac{1-\mu^2}{r}\frac{d}{d\mu};$$

whence, by (12),

$$\begin{aligned}4\pi\frac{d\psi}{dx}&=\mu P_n\frac{d\psi_n}{dr}+\frac{1-\mu^2}{r}\psi_n\frac{dP_n}{d\mu}\\&=-\iota\kappa\mu P_n\psi_{n+1}+\frac{n\psi_n}{r}P_{n-1}\\&=-\frac{\iota\kappa}{2n+1}\{n\psi_{n-1}P_{n-1}+(n+1)\psi_{n+1}P_{n+1}\}.\end{aligned}$$

Accordingly, (13) becomes

$$\frac{\iota\kappa\,dS}{4\pi(2n+1)}\{n\psi_{n-1}P_{n-1}+(2n+1)\psi_nP_n+(n+1)\psi_{n+1}P_{n+1}\}\ldots\ldots(14).$$

The effect of the element may, therefore, be represented by the combination of three multiple sources of orders $n-1$, n, $n+1$, where n is zero or any positive integer. If, therefore, the cause which produces the train of plane waves be removed, and the distribution of multiple sources given by (14) be substituted for it, the effect of this distribution at points lying on the positive side of the plane $x=0$ will be the same as that of the original train of waves, whilst the effect at points lying on the negative side will be zero.

If we put $n=0$, we shall fall back on our previous result. This result, therefore, gives the simplest way of resolving a plane wave, but it is only one out of an infinite number.

5. The preceding investigation renders it unnecessary to consider the corresponding optical problem in so much detail; but, inasmuch as the theory of singular points has proved of very great utility in a variety of branches of physics, it will be desirable to show that a similar theory exists in the case of elastic solids and other media whose motion is governed by similar equations.

The general equations of motion of an isotropic elastic solid are of the form

$$\frac{d^2u}{dt^2}=(a^2-b^2)\frac{d\delta}{dx}+b^2\nabla^2u\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$\&\text{c.},\quad \&\text{c.},$$

where a and b are the velocities of propagation of the dilatational and *distortional* waves respectively.

Assuming that all the quantities vary as $\epsilon^{\iota st}$, and putting $\kappa = s/b$, $h = s/a$, (15) becomes

$$(\nabla^2+\kappa^2)\,u+\left(\frac{\kappa^2}{h^2}-1\right)\frac{d\delta}{dx}=0 \quad\text{.....................(16).}$$

The solution of (16) is

$$u=\frac{d\phi}{dx}+u' \quad\text{................................ (17),}$$

where
$$(\nabla^2+h^2)\,\phi=0 \quad\text{..............................(18),}$$

and u', v', w' each satisfy an equation of the form

$$(\nabla^2+\kappa^2)\,u'=0 \quad\text{..............................(19),}$$

subject to the condition

$$\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=0 \quad\text{........................ (20).}$$

The function ϕ determines the dilatational or sound wave which we have already discussed. We must now consider the distortional or optical wave, which depends upon the quantities u', v', w'.

It is known that (19) are satisfied by

$$u'=r^{-n}\psi_n X_n, \quad v'=r^{-n}\psi_n Y_n, \quad w'=r^{-n}\psi_n Z_n,$$

where ψ_n is the function $r^{-1}\epsilon^{-\iota\kappa r}f_n(\iota\kappa r)$, and X_n, Y_n, Z_n are solid harmonics of positive degree n; and we have to determine these harmonics, so that u', v', w' shall satisfy (20). By means of a process similar to that employed by Lamb,* it can be shown that the required value of u' is

$$u=\frac{\psi_n}{r^n}\left(\frac{d\phi_{n+1}}{dx}+y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right)+\frac{(n+1)\,\psi_{n+2}\,r^{n+3}}{n+2}\,\frac{d}{dx}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \quad\text{...(21),}$$

with symmetrical expressions for v', w', where ϕ_n and χ_n are positive solid harmonics of degree n.

The right-hand side of (21) is easily seen to be a solution of (19), and by means of (5) and (6), and the known properties of solid harmonics, it can be verified by trial that these values satisfy (20).

6. The simplest solution is obtained by putting $n = 0$, in which case

$$\psi_0=r^{-1}\epsilon^{-\iota\kappa r}, \quad \phi_1=Ax+By+Cz, \quad \chi_0=\text{const.};$$

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* *Proc. Lond. Math. Soc.*, Vol. XIII., p. 51.

whence

$$u_0 = \frac{A\epsilon^{-\iota\kappa r}}{r} + \frac{\epsilon^{-\iota\kappa r}}{2r^3}\left\{1 + \frac{3}{\iota\kappa r} + \frac{3}{(\iota\kappa r)^2}\right\}\left\{Ar^2 - 3(Ax+By+Cz)\,x\right\} \qquad \text{.......... (22).}$$

This expression may be regarded as giving the value of u for a *simple source of light*, and it corresponds to a source in hydrodynamics, or to an electrified point. The expression is, however, more analogous to a doublet or a magnet, inasmuch as a simple source of light has *direction* as well as *magnitude*. The direction cosines of the axis of the source are proportional to A, B, C; and, if we suppose that its axis is parallel to z, we shall have $A = B = 0$. Also, at a considerable distance from the source, λ is usually so small in optical problems, in comparison with r, that powers of $(\kappa r)^{-1}$ may be neglected; whence, writing $F = \frac{3}{2}C$, we shall obtain

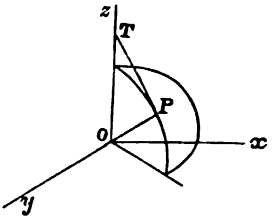
$$\left.\begin{aligned} u_0 &= -\frac{Fxz}{r^3}\,\epsilon^{-\iota\kappa r} \\ v_0 &= -\frac{Fyz}{r^3}\,\epsilon^{-\iota\kappa r} \\ w_0 &= \frac{F(x^2+y^2)}{r^3}\,\epsilon^{-\iota\kappa r} \end{aligned}\right\} \qquad \text{.................... (23).}$$

In the figure let P be the point x, y, z;

also let $\theta = POx$, $\phi = POz$;

draw PT perpendicular to OP in the plane POz. Then the preceding equations show that the direction of vibration is along PT, and its magnitude is equal to

$$\frac{F}{r}\,\epsilon^{-\iota\kappa r}\sin\phi.$$



Restoring the time factor $\epsilon^{\iota\kappa bt}$ and realising, this becomes

$$\frac{F}{r}\sin\phi\cos\frac{2\pi}{\lambda}(bt-r) \qquad \text{.................... (24).}$$

This is the expression for the disturbance produced by a simple source of light at a point r, whose distance from the source is large compared with the wave length; and it is of the same form as the expression obtained by Lord Rayleigh,* for the effect of a periodic force which acts at the origin and in the direction of the axis of z. The motion is, as might be expected, symmetrical with respect to the axis of z, and vanishes on that axis where $\phi = 0$ or π, and it is a maximum on the plane xy where $\phi = \tfrac{1}{2}\pi$.

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* *Phil. Mag.*, Feb., 1871.

7. In order to obtain the most general expression for a singular point of the second order, we must put $n = 1$; whence

$$\phi_2 = (B-C)\,x^2+(C-A)\,y^2+(A-B)\,z^2+2A'yz+2B'zx+2C'xy,$$

$$\chi_1 = ax+\beta y+\gamma z\,;$$

and

$$u_1 = \frac{\psi_1}{r}\left(\frac{d\phi_2}{dx}+\gamma y-\beta z\right)+\tfrac{2}{3}\psi_3 r^4\,\frac{d}{dx}\left(\frac{\phi_2}{r^5}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

The expression for a singular point of the second order accordingly contains eight constants, and is therefore a function of considerable generality. Let us now suppose, as a particular case, that

$$\phi_2 = \tfrac{3}{10}Fzx, \quad \chi_1 = \tfrac{1}{2}Fy\,;$$

then, if we confine our attention to points at a considerable distance from the origin, we may put

$$\psi_1 = \psi_3 = r^{-1}\epsilon^{-\iota\kappa r},$$

whence

$$\left.\begin{aligned} u_1 &= -\frac{Fx^2z}{r^4}\,\epsilon^{-\iota\kappa r} \\ v_1 &= -\frac{Fxyz}{r^4}\,\epsilon^{-\iota\kappa r} \\ w_1 &= \frac{Fx\,(x^2+y^2)}{r^4}\,\epsilon^{-\iota\kappa r} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

It therefore follows that the magnitude of the displacement represented by (26) is

$$\frac{F}{r}\,\epsilon^{-\iota\kappa r}\sin\phi\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

and that its direction is along PT. Restoring the time factor, adding (24) and (27), and writing $cdS/2\lambda$ for F, we obtain

$$\frac{cdS}{2\lambda r}\,(1+\cos\theta)\,\sin\phi\,\cos\frac{2\pi}{\lambda}\,(bt-r),$$

which is Stokes' result.

We therefore see that Stokes' expression for the disturbance produced by an element of a plane wave of light is equivalent to the combination of a simple and a double source, whilst Lord Rayleigh's expression* represents a simple source alone. It appears to me that Stokes' result is the preferable one of the two; partly for the reason that a distribution of simple sources of light, like simple sources of sound, produces a motion behind the wave, whilst the combination of a simple and double source in the above expression, as Stokes has shown, gives zero motion behind the wave; and also because the effect of an element upon a distant point must not only depend upon the direction of vibration (*i.e.*, upon ϕ), but also on the inclination of the line joining the point and the element with the plane of the latter. At the same time, if we were to carry out the investigation on the same lines as has been done in the case of sound at the commencement of this paper, there can, I think, be little doubt that we should find that there is an infinite number of combinations of multiple sources which would produce the required effect, and consequently Stokes' law, although the simplest, is only one out of an infinite number. The question is not, however, of very much importance in the case of light, inasmuch as, in problems relating to diffraction, we may with sufficient accuracy take $\sin\phi = \cos\theta = 1$, in which case the disturbance due to the element will be

$$\frac{c\,dS}{\lambda r}\cos\frac{2\pi}{\lambda}(bt-r),$$

corresponding to the wave

$$w = c\sin\frac{2\pi}{\lambda}(bt-x).$$

[8.† By means of the general solution (21) combined with (17) which gives the dilatational wave, it would be possible to work out the problem of a spherical pendulum vibrating in ether, and also the scattering of a plane wave of light by a fixed or movable spherical obstacle. But, before attacking any problem of this nature, it would be first of all necessary to consider the boundary conditions.

With regard to these conditions three suppositions may be made.

1st. We may suppose that no slipping takes place, which requires that the velocity of the ether in contact with the sphere should be equal to that of the sphere itself; but, inasmuch as there are reasons for thinking that the amplitudes of the vibrations of the matter are very much smaller than those of the ether in contact with it, except in the extreme case in which one of the free periods of the matter is equal to the period of the ethereal waves, this hypothesis is improbable.

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* *Encycl. Brit.*, Art. "Wave Theory," p. 453, equation (20).

† Rewritten June 14th, 1891.

2nd. We may suppose that partial slipping takes place. This hypothesis is open to the objection that the law of slipping is unknown, and would therefore involve an additional assumption; and also that it would introduce frictional resistance. It might, at first sight, appear that frictional resistance might be explained by supposing that there is a conversion of energy into heat; but, in attempting to construct a molecular theory, we ought to regard radiant heat as consisting of ethereal waves of longer period than waves of light.

3rd. We may suppose that perfect slipping takes place. In this case the boundary conditions are continuity of normal motion, and zero tangential stress. This hypothesis has much to commend it on the ground of simplicity, since the action of the ether on the matter consists of a hydrostatic pressure, and in the case of a sphere is consequently reducible to a force; whereas, if no slipping or partial slipping took place, the action would (except in special cases) consist of a couple as well as a force.

The solution of the problems suggested would undoubtedly throw light on the phenomena of absorption and anomalous dispersion; but the question of absorption can be illustrated just as well, and more simply as far as analysis is concerned, by considering the corresponding acoustical problem.]

9. The theory of absorption, which was first suggested by Stokes,[*] and afterwards amplified and more fully developed by Sellmeier,[†] depends upon the theorem, that when a dynamical system is subjected to a force whose period is equal, or nearly so, to one of the free periods of the system, the amplitude of the corresponding forced vibration will be large.[‡] In certain cases, the amplitude of the forced vibration

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[*] *Phil. Mag.*, March 1860, p. 196.

[†] *Pogg. Ann.*, Vol. CXLV., pp. 399, 520; CXLVII., pp. 386, 525.

[‡] [It must be recollected that Herschel's theorem, that the period of the forced vibration is equal to that of the force, is only true when the molecular forces depend upon the first powers of displacements. If, for example, the equation of motion were of the form

$$\frac{d^2y}{dt^2}+\mu^2y^3=\frac{2p^3}{3\mu\sqrt{3}}\cos 3pt,$$

becomes infinite when the periods are equal; and, in order to avoid this difficulty, Helmholtz[*] in his theory of anomalous dispersion introduced a viscous term (*i.e.*, a term proportional to the velocity), which he endeavoured to account for on the supposition that it represented a conversion of energy into heat. The introduction of this term is objected to by Sir W. Thomson,[†] on the ground that we have no right to suppose that the luminiferous ether is a medium which is capable of exerting any action in the nature of frictional resistance; but, on the other hand, Sir W. Thomson's molecular theory is open to the objection which Helmholtz sought to evade, inasmuch as the amplitudes of the vibrations of his molecules and the index of refraction become infinite when the period of the waves of light is equal to any one of the free periods of the matter. It will therefore be desirable to show that the viscous term introduced by Helmholtz may be accounted for on physical grounds, which are not open to Sir W. Thomson's objections.

If plane waves of sound whose period is τ impinge upon a spherical obstacle, whose radius is small in comparison with the wave-length of the waves of sound, and if the sphere be attached to a spring so as to be capable of vibrating parallel to the direction in which the waves are travelling, it can be shown that the effect of the waves will be to cause the sphere to perform small oscillations. The energy of the sphere is supplied in the first instance by the waves which impinge upon it, and the motion of the sphere generates divergent waves which travel away into space carrying energy with them. If, therefore, the incident waves ceased, the sphere would

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the forced vibration would be $y = \frac{2p}{\mu\sqrt{3}}\cos pt$;

whence the period of the forced vibration is three times that of the force. An equation of this kind illustrates the way in which ultra-violet waves might be converted into luminous waves. See Stokes, "On the Change of Refrangibility of Light," *Phil. Trans.*, 1852, p. 463.

The complementary function is

$$y = a \operatorname{cn}(a\mu t + \alpha), \quad k = 2^{-\frac{1}{2}},$$

where a and α are the constants of integration. The amplitude is a, and the period $4K/\mu a$. This result is an example of a vibration whose period depends upon its amplitude. If, therefore, we were to suppose that waves of dark heat falling upon a body were capable of producing vibrations of this form, in which a increases slowly with the time, the body would first become hot, then red-hot, and afterwards intensely luminous.—*Added June 14th*, 1891.]

[*] *Wissen. Abhand.*, Vol. II., p. 213.

[†] "Lectures on Molecular Dynamics," p. 149.

gradually get rid of all its energy by generating scattered waves, and would ultimately come to rest; hence the equation of motion of the sphere must contain a viscous term, notwithstanding the fact that the system is assumed to be devoid of viscosity.

Let τ be the period of the sound waves, τ' the free period of the sphere, M the mass of the latter, and M' the mass of the fluid displaced; also let the radius c of the sphere be so small in comparison with the wave-length of the sound waves, that powers of κc higher than the cube may be neglected. Then it can be shown that the equation of motion of the sphere is

$$\left\{M+\tfrac{1}{4}M'(2+\kappa^2c^2)\right\}\ddot{\xi}+\frac{M'\pi}{2\tau}\kappa^3c^3\dot{\xi}$$

$$+\frac{4\pi^2M'}{a\tau^2}\left(\tfrac{3}{2}+\tfrac{3}{4}\iota\kappa^3c^3\right)\epsilon^{2\iota\pi t/\tau}+\frac{4\pi^2M}{\tau'^2}\xi=0,$$

which puts in evidence the existence of the viscous term.

The amplitude A is determined by the equation

$$A=\frac{M'\tau'^2\left(\tfrac{3}{2}+\tfrac{3}{4}\iota\kappa^3c^3\right)/a}{M\left(\tau'^2-\tau^2\right)+M'\left(\tfrac{1}{2}+\tfrac{1}{4}\kappa^2c^2\right)\tau'^2-\tfrac{1}{4}\iota\kappa^3c^3M'\tau'^2}.$$

Since this expression contains an imaginary term in the denominator, it follows that the amplitude can never become infinite for any real value of τ.

For all solids M is very much greater than M'; hence, if the difference between τ and τ' be not very small, A will be very small; but, if $\tau=\tau'$, the large term in the denominator involving M disappears, and A is approximately equal to $3/a$.

To the above order of approximation, the amplitude will be greatest

when $$\tau^2=\tau'^2+\frac{M'\tau'^2}{2M}\left(1+\tfrac{1}{2}\kappa^2c^2\right),$$

in which case $$A=-\frac{3}{a}\left(1+\frac{2}{\iota\kappa^3c^3}\right),$$

which is large on account of the smallness of κ^3c^3.

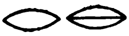
Yoke-Chains and Multipartite Compositions in connexion with the Analytical Forms called "Trees."

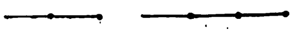
By Major P. A. MacMahon, R.A., F.R.S.

[*Read April 9th*, 1891.]

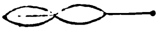
A yoke-chain, an expression adopted at the suggestion of Professor Cayley, is a geometrical configuration composed of line branches.

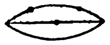
The simple line branch —— is indifferently a yoke or a chain.

The combinations  &c. are yokes;

whilst  &c. are chains;

and generally we form chains by combining chain-wise any number of yokes, and also generally we form yokes by combining yoke-wise any number of chains.

E.g.,  is a chain,

 is a yoke.

Yoke-chains may be viewed as diagrammatic representations of the combinations of resistances of linear electrical conductors, or of the capacities of electrical condensers. The yoke and the chain represent parallel and series combinations of resistances, but series and parallel combinations of capacities.

The theory of yoke-chains does not include every combination of resistances, but only those which are made up of two or more combinations, either in parallel or in series. In the case of five resistances, for example, we find a combination such as the Wheatstone net, which is not decomposable into other combinations either in parallel or in series. Such networks, considered by Kirchhoff and Maxwell, do not come into view here.

Consider in the first place the different yoke-chains that can be formed from a given number of branches.

From a single branch we can merely form the yoke or chain.

From two branches we form the chain of Fig. 1 or the yoke of Fig. 2,

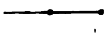


Fig. 1.

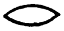


Fig. 2.

I define the yoke of Fig. 2 to be the yoke-chain conjugate to the chain of Fig. 1, and I further regard the yoke or chain —— as being self-conjugate.

Every chain is a chain of yokes, and every yoke is a yoke of chains.

I make the following definitions :—

Definition.—The conjugate of any chain is formed by placing the conjugates of the component yokes in a yoke.

Definition.—The conjugate of any yoke is formed by placing the conjugates of the component chains in a chain.

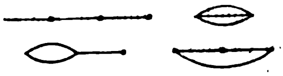
The process of conjugation is necessarily reversible, as is obvious from the definitions.

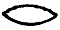
The second of the above definitions is derived from the first by interchanging the words chain and yoke.

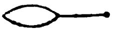
We have thus the notion of conjugate yoke-chains, and it is clear that the yoke-chains of a given order (*i.e.*, of a given number of branches) may be arranged in conjugate pairs, each pair comprising a chain and a yoke.

Further, the whole of the yoke-chains may be arranged in a chain set and a yoke set, and either set is derivable by conjugation from the other.

Passing now to the case of three branches, we form the chain set by placing each of the forms of order 2 in chain with the form of order 1. We thus obtain the left-hand column below :



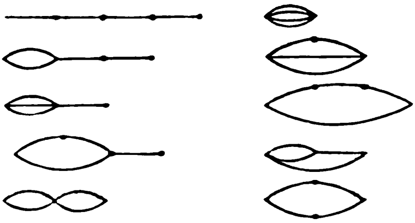
To form the right-hand column, we first place the three self-conjugate components of the form ——•——• in yoke, and then place the conjugates of the components  —— of the form



viz., ——• and •——, in yoke.

We thus obtain the left-hand chain set, and the right-hand yoke set.

In forming the forms of order 4, we place each of the four forms of order 3 in chain with the form of order 1, and further place the yoke form of order 2 in chain with itself. We have thus $5\,(=4+1)$ forms, constituting the chain-column of order 4. These are given below, and also the conjugate yoke-column.



To form the chain-column of order 5, each of the ten forms of order 4 is placed in chain with the form of order 1, and further each of the yoke forms of order 3 is placed in chain with the yoke form of order 2. There are thus, in all, $2\,(10+2) = 24$ forms. In general, the forms of order n are formed from the forms of lower orders. To form the chain-column, as many processes must be performed as there are non-unitary partitions of n, or as there are partitions of n composed of the integers

$$2,\ 3,\ 4,\ \ldots\ n.$$

We first place each form of order $n-1$ in chain with the form of order 1. This is the first process, and thereafter we have a process associated with every non-unitary partition of n with the exception of the partition consisting merely of the number n itself.

For the partition $(n-2, 2)$ we take every yoke form of order $n-2$ in chain with the yoke form of order 2. For the partition $(n-p, p)$, $p>1$, we take every yoke form of order $n-p$ in chain with every yoke form of order p.

Denote by B_p the number of chain or of yoke forms of order p. Then, if $p = 1$, the complete number of forms is $B_1 = 1$: but if $p>1$ the complete number of forms is $2B_p$.

In forming synthetically the number B_n we have above found a portion $2B_{n-1}$, and corresponding to the partition $(n-p, p)$, $p>1$, a portion $B_{n-p}B_p$, for this represents the number of ways of combining one of $n-p$ things with one of p things of a different sort.

If, however, $n-p=p$, the corresponding portion is not B_p^2, but $\frac{B_p(B_p+1)}{2}$, viz., the number of homogeneous products of B_p things, two and two together. So also, corresponding to the partition $(l^{\lambda} m^{\mu} \ldots)$, we have the number

$$\frac{B_l(B_l+1)\ldots(B_l+\lambda-1)}{\lambda!} \cdot \frac{B_m(B_m+1)\ldots(B_m+\mu-1)}{\mu!} \ldots,$$

and finally we have the relation

$$B_n = 2B_{n-1}+B_{n-2}B_2+B_{n-3}B_3+\ldots$$
$$\ldots+\Sigma\frac{B_l(B_l+1)\ldots(B_l+\lambda-1)}{\lambda!} \cdot \frac{B_m(B_m+1)\ldots(B_m+\mu-1)}{\mu!} \ldots,$$

the numbers l, m, &c., ... being >1.

This relation leads at once to the expression of the law of the numbers B in the form

$$(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$=1+(B_1-1)x+(2B_2-B_1)x^2+2(B_3-B_2)x^3+\ldots+2(B_p-B_{p-1})x^p+\ldots,$$

or, multiplying up by $1-x$,

$$(1-x)^{-B_1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$=1+B_1x+2(B_2x^2+B_3x^3+B_4x^4+\ldots).$$

This formula, wherein $B_1 = B_2 = 1$, is convenient for calculating the numbers $2B_p$.

We find

B_1	$2B_2$	$2B_3$	$2B_4$	$2B_5$	$2B_6$	$2B_7$	$2B_8$	$2B_9$	$2B_{10}$ ...
1	2	4	10	24	66	180	522	1532	4984 ...

There is a paper by Cayley in the *Philosophical Magazine*, Vol. XIII. (1857), also *Collected Papers*, Vol. III., No. 203, "On the Theory of the Analytical Forms called Trees." The following passage occurs (*Collected Papers*, *loc. cit.*, pp. 245, 246)* :—

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* Other references are—

"On the Analytical Forms called Trees," Cayley, *Phil. Mag.*, Vol. XX. (1860), pp. 337–341.

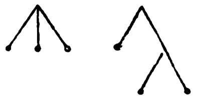
"On the Mathematical Theory of Isomers," Cayley, *Phil. Mag.*, Vol. XLVII. (1874), p. 444.

"On the Analytical Forms called Trees, with Application to the Theory of Chemical Combinations," Cayley, *British Association Report*, 1875, pp. 257–305.

"On the Analytical Forms called Trees," Cayley, *American Journal of Mathematics*, Vol. IV., pp. 266–268.

In the latter paper the notion of the centre or bicentre of number is stated to be due to M. Camille Jordan, but I cannot find the reference.

"I have had occasion for another purpose to consider the question of finding the number of trees with a given number of free branches, bifurcations at least. Thus, when the number of free branches is three, the trees of the form in question are those in the annexed figure, and the number is therefore two. It is not difficult to see that



we have in this case (B_r being the number of such trees with r free branches)

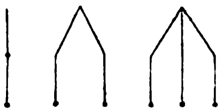
$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\dots$$
$$= 1+x+2B_2x^2+2B_3x^3+2B_4x^4+\&c."$$

In view of this interesting identity of enumeration, though in the absence of information in regard to the purpose for which Cayley investigated the subject, I propose to examine in detail the correspondence between yoke-chains and the trees with free branches.

A single tree represents either member of a certain conjugate pair of yoke-chains, according to the interpretation placed upon the combination of knots and branches which constitutes the tree.

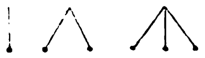
In the first place, we may restrict attention to the chain combinations of the several orders.

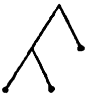
One, two, three, &c. branches in chain may be denoted by the trees



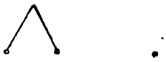
and this representation would be in some respects the most consistent with what follows. The idea is that a branch

denotes a chain of order 1, and the upper knots are then joined by branches to a single knot to denote that the several chains are to be joined in chain. The trees, however, become simplified if we agree to represent a single linear resistance, not by a branch but by a terminal knot. The three trees above then become

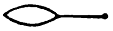


Consider next the tree 

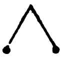
which may be taken to denote the resistances

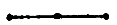


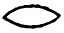
placed in series. In order that this tree may represent the combination

,

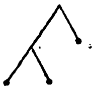
it is necessary to suppose that the tree



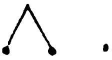
represents not the combination  but rather its conjugate yoke form

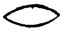
.

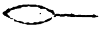
The ultimate step in interpreting a tree will be to take a number of forms in chain. The trees which represent these forms will be found pendent to the second row of knots in the tree. We must therefore agree always to interpret the trees which originate from the second row of knots as yoke combinations. On this convention the tree



denotes the chain combination of the yoke combinations denoted by the trees



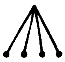
viz., the yokes  ——, placed in chain, or

.

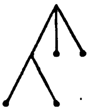
The above principle enables us to form in succession the free-branch trees of the various orders.

The five trees of order 4 are

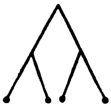
I.



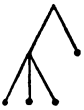
II.



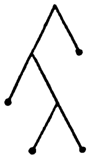
III.



IV.

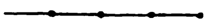


V.

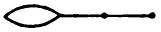


Each corresponds to some partition of four other than the number 4 itself.

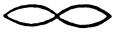
The tree I. [partition (1^4)] clearly denotes

 ;

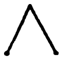
the tree II., partition (21^2), denotes

 ;

the tree III., partition (2^2), gives

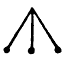
 ;

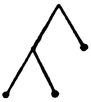
trees IV. and V. both belong to the partition (31); this follows from the fact that there are two trees of the order 3, and either may be placed as a pendent to a knot of the tree



in order to form a tree of order 4.

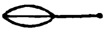
According to the rule, we must give parallel interpretations to the trees

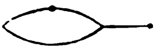




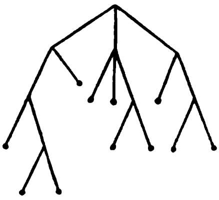
prior to taking them in series, in each case, with the tree denoted by a single knot.

We have thus—

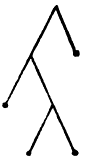
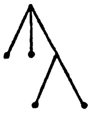
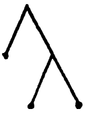
Tree IV. ≡ 

Tree V. ≡  and the five trees have been placed in correspondence with the five series combinations of order 4.

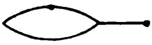
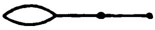
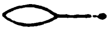
As another example I take the tree



The series equivalences of the trees

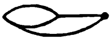
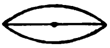
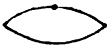




are

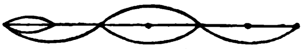




To form the complete combination we have merely to take the conjugates of these in series.

The conjugates are

so that the tree denotes the combination

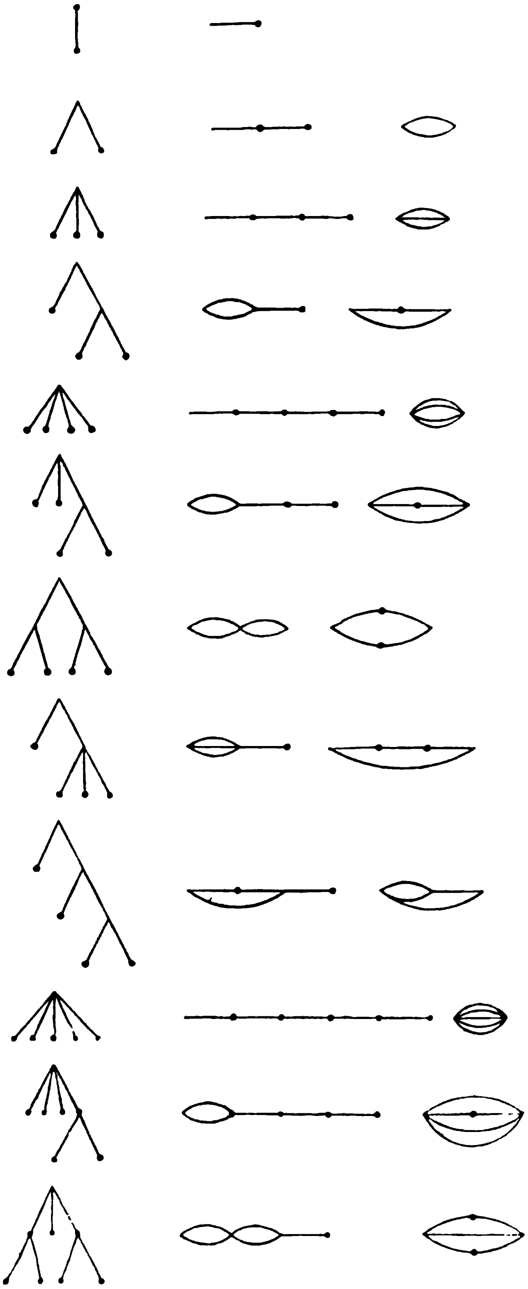


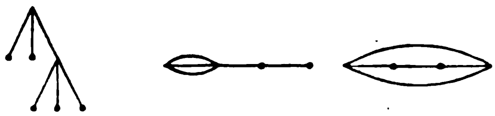
In this way any tree may be interpreted so as to denote a chain.

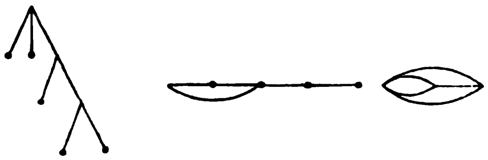
If in the foregoing rules the words chain and yoke be interchanged, the result will be the conjugate yoke combination.

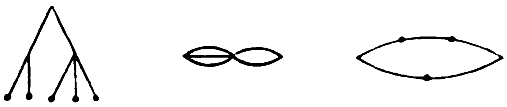
A tree therefore may be taken to be a representation of either combination of a conjugate pair at pleasure, or if we please of both combinations of such a pair.

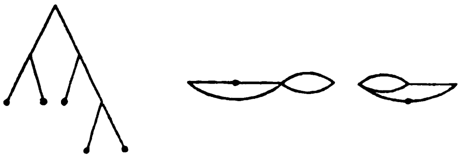
I give below a Table showing the correspondence of trees with yoke-chains of the first six orders.

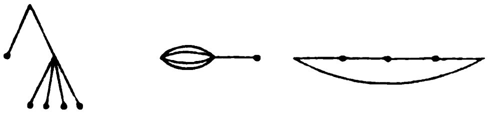


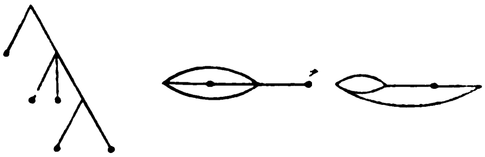


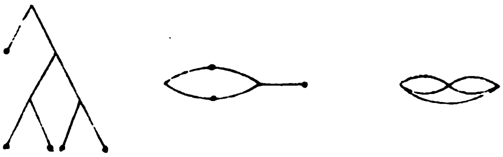


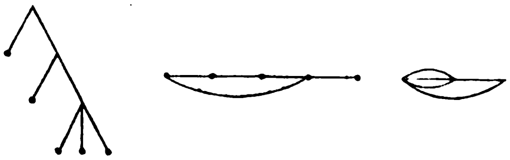


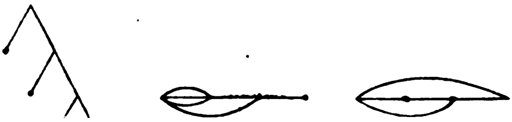












The capacity of condensers placed in parallel is formed according to the same law as the resistance of linear conductors placed in series.

In general the capacity of any combination of condensers is formed according to the same law as the resistance of the conjugate combination of linear conductors.

There is a very simple result connected with the conjugate combinations of equal linear conductors which may possibly have escaped notice.

If any combination of linear conductors, each of r ohms resistance, be formed so that the combined resistance is

$$\frac{m}{n}r \text{ ohms};$$

then the resistance of the conjugate combination is

$$\frac{n}{m}r \text{ ohms}.$$

The inductive proof is easy, for suppose two series combinations to have resistances

$$\frac{a}{b}r \text{ and } \frac{a'}{b'}r \text{ ohms},$$

and to be such that the resistances of the conjugate combinations are

$$\frac{b}{a}r \text{ and } \frac{b'}{a'}r \text{ ohms}.$$

Placing the two combinations in parallel, the resistance is

$$\frac{\frac{aa'}{bb'}}{\frac{a}{b}+\frac{a'}{b'}}r \text{ ohms},$$

which is

$$\frac{1}{\frac{b}{a}+\frac{b'}{a'}}r \text{ ohms},$$

and the reciprocal of the multiplier of r is

$$\frac{b}{a}+\frac{b'}{a'},$$

which, multiplied into r, represents the resistance of the combination formed by placing the conjugates of the component combinations in series.

Hence, since the law evidently holds in the simplest cases, it must be true in general.

2. In the *Philosophical Magazine*, 1860, Professor Cayley has enumerated the trees with a given number of terminal knots. He remarks :—

"We have here

$$\phi m = 1 \,.\, 2 \,.\, 3 \dots (m-1) \text{ coefficient } x^{m-1} \text{ in } \frac{1}{2-\exp x},$$

giving the values

$$\phi m = 1,\ 1,\ 3,\ 13,\ 75,\ 541,\ 4683,\ 47293,\ \dots,$$

for

$$m = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \dots."$$

This enumeration is identical with that of the compositions of certain multipartite numbers.[*]

The correspondence is between the trees with m terminal knots, and the compositions of the multipartite number

$$\overline{1^{m-1}},$$

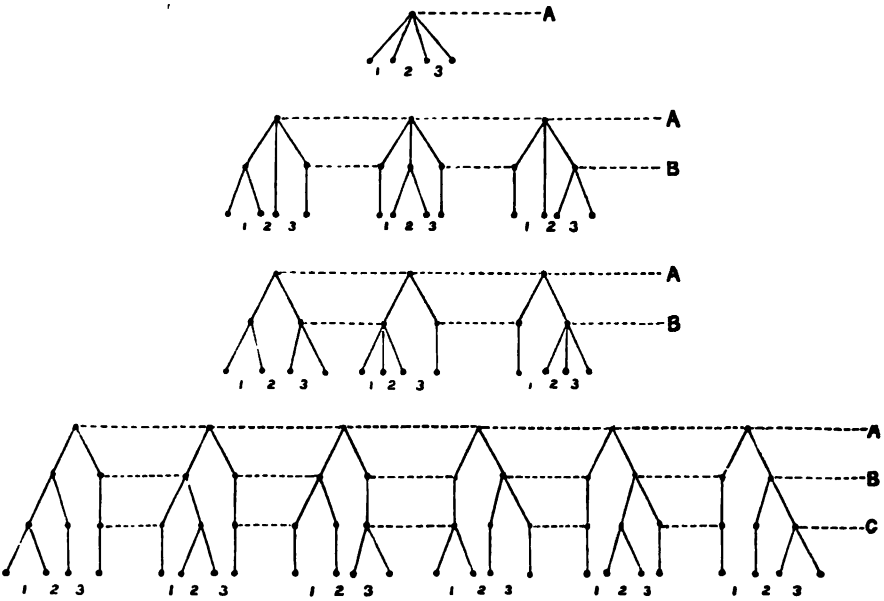
where 1^{m-1} denotes 1 repeated $m-1$ times, and the bar — means that the partition does not denote the partition (1^{m-1}) of a single number $m-1$, but rather the multipartite number having the multipartite weight 1, 1, 1, 1, ..., $m-1$ times.

To identify each tree with a composition, consider, for example, the 13 trees with 4 terminal knots.

---

[*] H. J. S. Smith and J. W. L. Glaisher have termed partitions in which the order of the parts is essential "compositions." For multipartite numbers see the author, "Memoir on Symmetric Functions of the Roots of Systems of Equations," *Phil. Trans. R. S. of London*, Vol, CLXXXI. (1890), A., pp. 481–536; and for compositions of multipartite numbers, the author, the *Messenger of Mathematics*, Vol. XX. (1890).

These are



We have to identify each of these trees with a composition of the tripartite number $\overline{111}$. The number being tripartite, whilst the terminal knots are four in number, the idea is presented of considering not the terminal knots, but the spaces between them, which are but three in number. These spaces are numbered 1, 2, 3 from left to right in each tree. Consider, moreover, the knots in the first, second and third rows, omitting the terminal row of knots. These are marked A, B, C; in each row we have knots from which descend two or more branches. Of these branches any two that are adjacent bound an area which is in direct communication with one of the numbered spaces between the terminal knots. In any row of knots we must observe the connexion between each angle, formed by the

descent of two adjacent branches from a knot, and the terminal spaces.

In the second of the last line of trees, beginning with row A, we see an angle in communication with space 3, but not with spaces 1 or 2; this connexion between angles and spaces may be denoted by

001 ;

similarly for row B, there is one angle in connexion with space 1, so that we have

$$\overline{100};$$

in row C, one angle leads to space 2, so that we have

$$\overline{010};$$

the whole connexion between angles and spaces is represented by the composition

$$(\overline{001},\ \overline{100},\ \overline{010})$$

of the multipartite number $\overline{111}$.

Interpreting each tree in succession, we get the whole of the 13 compositions of $\overline{111}$, viz.:—in order

$$(\overline{111}),$$

$$(\overline{011},\ \overline{100}),\quad (\overline{101},\ \overline{010}),\quad (\overline{110},\ \overline{001}),$$

$$(\overline{010},\ \overline{101}),\quad (\overline{001},\ \overline{110}),\quad (\overline{100},\ \overline{011}),$$

$$(\overline{001},\ \overline{010},\ \overline{100}),\quad (\overline{001},\ \overline{100},\ \overline{010}),\quad (\overline{010},\ \overline{100},\ \overline{001}),$$

$$(\overline{010},\ \overline{001},\ \overline{100}),\quad (\overline{100},\ \overline{001},\ \overline{010}),\quad (\overline{100},\ \overline{010},\ \overline{001}).$$

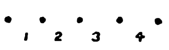
This principle of interpretation is perfectly general. The number of angles in a tree must be less by one than the number of terminal knots, and each row from the first (or top) to the penultimate must contain at least one angle.

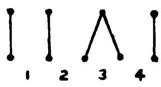
If any composition of a multipartite number of the form which presents itself in this theory be given, it is extremely easy to form the corresponding tree. To form the tree, it is best to commence with the right-hand part of the composition, and to then proceed regularly towards the left.

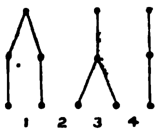
Suppose given the composition

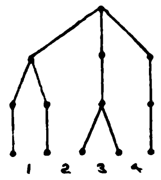
$$(\overline{0101},\ \overline{1000},\ \overline{0010}),$$

the successive operations are

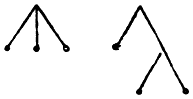








"I have had occasion for another purpose to consider the question of finding the number of trees with a given number of free branches, bifurcations at least. Thus, when the number of free branches is three, the trees of the form in question are those in the annexed figure, and the number is therefore two. It is not difficult to see that



we have in this case (B_r being the number of such trees with r free branches)

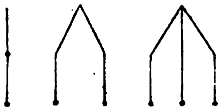
$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$= 1+x+2B_2x^2+2B_3x^3+2B_4x^4+\&c."$$

In view of this interesting identity of enumeration, though in the absence of information in regard to the purpose for which Cayley investigated the subject, I propose to examine in detail the correspondence between yoke-chains and the trees with free branches.

A single tree represents either member of a certain conjugate pair of yoke-chains, according to the interpretation placed upon the combination of knots and branches which constitutes the tree.

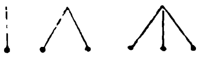
In the first place, we may restrict attention to the chain combinations of the several orders.

One, two, three, &c. branches in chain may be denoted by the trees

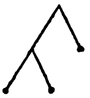


and this representation would be in some respects the most consistent with what follows. The idea is that a branch

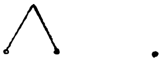
denotes a chain of order 1, and the upper knots are then joined by branches to a single knot to denote that the several chains are to be joined in chain. The trees, however, become simplified if we agree to represent a single linear resistance, not by a branch but by a terminal knot. The three trees above then become



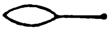
Consider next the tree



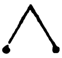
which may be taken to denote the resistances

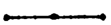


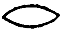
placed in series. In order that this tree may represent the combination

,

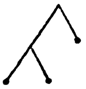
it is necessary to suppose that the tree



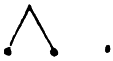
represents not the combination  but rather its conjugate yoke form

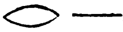
.

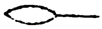
The ultimate step in interpreting a tree will be to take a number of forms in chain. The trees which represent these forms will be found pendent to the second row of knots in the tree. We must therefore agree always to interpret the trees which originate from the second row of knots as yoke combinations. On this convention the tree



denotes the chain combination of the yoke combinations denoted by the trees



viz., the yokes , placed in chain, or

.

The above principle enables us to form in succession the free-branch trees of the various orders.

we reach without difficulty the formula

$$T_{1,n}x+T_{2,n}x^2+T_{3,n}x^3+\ldots=\frac{x}{1-nx}.$$

Hence, as above stated, $T_{m,n}=n^{m-1}$.

We have established the theorem :—

"The number of compositions of the multipartite number

$$\overline{1^{m-1}}$$

into n parts, zeros not excluded, is

$$n^{m-1},$$

and there is a one-to-one correspondence between these compositions and the trees of altitude n which have m terminal knots."

On Functions determined from their Discontinuities and a certain form of Boundary Condition. By Mr. W. Burnside.

[*Read May* 14*th*, 1891.]

The problem of uniform streaming in two dimensions (of incompressible fluid or electricity) may be stated in the following form: to determine a single-valued function of $x+iy$ which in a given region of the x, y plane shall have a single given infinity and whose imaginary part shall be constant over the boundary of the region.

It is the object of the following paper to prove certain general theorems with respect to functions of a complex variable which are determined by conditions similar to but more general than these, and to show how to construct the functions themselves in certain simple cases.

I suppose that m closed non-intersecting curves are given, $C_1, C_2 \ldots C_m$, no one of which separates any two others. The region bounded by these curves is called R; and I consider a function w in the region R which shall be everywhere one-valued and continuous, except at n given points $z_1, z_2 \ldots z_n$, at each of which it has a given simple infinity (so that $w-\dfrac{a_r}{z-z_r}$ is finite when $z=z_r$), and which at the boundaries shall be such that the imaginary part of $we^{i\theta_r}$ is constant over C_r, where $\theta_1, \theta_2 \dots \theta_m$ are given angles.

Assuming the function w to exist, I prove that it is completely determined, except as regards an additive constant, by the given conditions, and that in R it takes every value n times. It follows at once that if $w_1, w_2 \dots w_n$ are functions satisfying the same boundary conditions and each with a single infinity, then

$$w = w_1 + w_2 + \dots + w_n.$$

I then go on to show how to form the functions when the boundaries are circles.

First consider the nature of the condition at the boundaries. If

$$w = u + iv,$$

then, at C_r, $u \sin \theta_r + v \cos \theta_r$ is constant. Hence

$$\frac{\partial u}{\partial s} \sin \theta_r + \frac{\partial v}{\partial s} \cos \theta_r = 0,$$

if $\dfrac{\partial}{\partial s}$ denote differentiation along the arc of C_r. But $\dfrac{\partial}{\partial n}$ denoting differentiation along the normal to C_r,

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n} \text{ and } \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s};$$

and if at any point on C_r $\dfrac{\partial u}{\partial s}$ vanishes then also $\dfrac{\partial u}{\partial n}$ vanishes, and the point is a double point on the u-curve passing through it.

Now u and v cannot both be constant along C_r without w being everywhere constant; hence there must be an even number of points on C_r at which u and v have (as regards displacements on C_r) maximum or minimum values, and at which the curves $u =$ constant and $v =$ constant have double points. Suppose now that there are two different functions w' and w'' satisfying the given conditions in and at the boundaries of R; their difference $w' - w''$ $(= w)$ will be a function everywhere one-valued and continuous within R, and such that

$$u \sin \theta_r + v \cos \theta_r$$

is constant over C_r, where the suffix may have any value from 1 to m.

Suppose A and B are two points on C_1, at which

$$u = u_0.$$

The curve $$u = u_0,$$

which may, without loss of generality in the reasoning, be supposed not to extend to infinity, will consist of a branch starting from A and abutting on some other part of the boundary C_i, say at A_i; leaving C_i at another point B_i, and abutting on C_j; and so on, at last returning to B. Between A and B on C_1 is a point at which the u-curve, say

$$u = u_1,$$

has a double point. The curve

$$u = u_1$$

will either not meet C_i, or abut on it at two points between A_i and B_i, and a u-curve drawn for some value which u takes twice between A_i and B_i, will certainly abut on one bounding curve less than

$$u = u_0.$$

This process may be continued, and at last there must be a curve

$$u = u_r,$$

which abuts only on one bounding curve C_s. But between the two points on C_s where

$$u = u_r$$

meets it, there is one at which the u-curve has a double point. This u-curve then must either cut

$$u = u_r,$$

which is contrary to the supposition that w is one-valued, or it must be closed, which is contrary to the supposition that w is everywhere finite. Hence the supposition that $w'-w''$ is not constant leads to a contradiction. The function then, if it exists, is completely determined, except as regards an additive constant, by the given conditions.

For sufficiently great values of u_0 and v_0, the curves

$$u = u_0 \quad \text{and} \quad v = v_0$$

consist each of n small closed approximately circular curves passing through $z_1, z_2 \dots z_n$ in pairs, and each pair intersecting in one other point.

Hence, when u_0 and v_0 are great enough, w takes the value u_0+iv_0 n times. Now following the reasoning that Prof. F. Klein applies to the similar problem for a one-valued function on a Riemann's surface ("Ueber Riemann's Theorie der Algebraischen Functionen," p. 46),

the curves $u = u_0, \quad v = v_0,$

can by continuous variation of u_0 and v_0 only lose or gain an intersection at a double point on the u- and v- curves or at the boundary. But as u_0 and v_0 change continuously through values corresponding to a double-point, a pair of points of intersection approach, coincide, and then separate into two again, so that no point of intersection is gained or lost in this way. Also, if

$$u = u_0$$

abuts on a boundary at A and B, the values of v at A and B are the same (from the boundary condition); and, since between A and B there must be a double point, the values of v on the curve

$$u = u_0,$$

as it approaches A and leaves B, must be either continually increasing or continually decreasing. Hence, as regards intersections, there is no difference between a curve that meets the boundary and one that does not. It follows therefore that the number of points of intersection is always the same as when u_0 and v_0 are very great; or in other words, w takes every value n times in the region R.

If w_1, $w_2 \dots w_n$ are n functions, each with a single infinity in the region R, and all satisfying the same boundary conditions as before, then $w_1+w_2+\dots+w_n$ is a single-valued function in R with n given infinities and satisfying the same boundary conditions; but it has just been seen that there is only one such function, and therefore any function of the kind considered can be formed by combining linearly functions with a single infinity. The analogy between the functions w_1, w_2, &c., in the region R, and linear functions of z in the infinite plane is obvious.

If w_1 is the function with a single infinity at z_1, the equation

$$w_1 = f_1(z),$$

establishes the conformable representation of the region R on the infinite w-plane, the bounding curves S_1, S_2, &c., corresponding to the two sides of finite portions of the lines

$$u \sin\theta_1 + v \cos\theta_1 = c_1,$$

$$u \sin\theta_2 + v \cos\theta_2 = c_2,$$

&c.,

where c_1, c_2, &c. are constants depending on the bounding curves and on z_1; and similarly

$$w = f(z)$$

establishes the conformable representation of R on an n-sheeted Riemann's surface, in which m finite lengths of certain straight lines are to be regarded as boundaries.

Now if in R a closed curve be drawn that does not surround one of the curves C, it can be reduced to a point by continuous deformation without leaving R. Hence the same must be true of the above Riemann's surface; or, in other words, the surface itself, leaving the finite bounding lines out of account, must be simply connected. Since it has n sheets, there must therefore be $2(n-1)$ branch-points; and therefore within the region R (*i.e.*, excluding the boundary), there must be $2n-2$ double points on the u- and v-curves.

Now suppose that, on S_r, u, and therefore v, has p_r maxima and p_r minima. Then I have shown, in a paper in the *Messenger of Mathematics* (Vol. XX., p. 66), that the whole number of double-points of the u- or the v-curves in R, including those at the boundaries, is

$$2n+\Sigma p_r+m-2.$$

But it has been shown here that in this case there are $2n-2$ double points inside R and $2\Sigma p_r$ on the boundaries.

Therefore $$\Sigma p_r = m;$$

but p_r cannot be less than unity, and hence

$$p_1 = \dots = p_r = \dots = p_n = 1;$$

or u and v will each have one maximum and one minimum at each boundary.

I go on now to consider the actual formation of the functions when the boundaries are circles.

With a single circle the solution is simple and well-known; but it will be convenient to express the result directly in this case for the sake of explaining the notation to be used in the further cases.

The conjugate imaginary of z will always be represented by z', the modulus of z by $|z|$, and the radii and centres of the various circles by r_1, z_1; r_2, z_2, &c. For a single circle the function w must satisfy the following conditions:—

$$w-\frac{A}{z-a}$$

must be finite for all points on the same side of the circle

$$|z-z_1| = r_1$$

as a, and the imaginary part of $we^{i\theta_1}$ must be constant at the circle.

Now, it is easy to verify that the imaginary part of

$$\frac{Ae^{i\theta_1}}{z-a}-\frac{A'e^{-i\theta_1}}{a'-z_1'-\dfrac{r_1^2}{z-z_1}}$$

vanishes at the circle

$$|\,z-z_1\,|=r_1,$$

for, writing

$$z-z_1=r_1e^{i\theta},$$

the expression becomes

$$\frac{Ae^{i\theta_1}}{r_1e^{i\theta}+z_1-a}+\frac{A'e^{-i\theta_1}}{r_1e^{-i\theta}+z_1'-a'},$$

the sum of two conjugate imaginaries, which is real.

Hence
$$w=\frac{A}{z-a}-\frac{A'e^{-2i\theta_1}}{a'-z_1'-\dfrac{r_1^2}{z-z_1}}+\text{constant}.$$

But
$$\frac{1}{a'-z_1'-\dfrac{r_1^2}{z-z_1}}=\left(\frac{r_1}{a'-z_1'}\right)^2\frac{1}{z-z_1-\dfrac{r_1^2}{a'-z_1'}}+\frac{1}{a'-z_1'},$$

and $z_1+\dfrac{r_1^2}{a'-z_1'}$, is the inverse point of a in the circle $|\,z-z_1\,|=r_1$, which will always be represented by a_1; therefore

$$w=\frac{A}{z-a}-\left(\frac{r_1}{a'-z_1'}\right)^2\frac{A'e^{-2i\theta_1}}{z-a_1}+\text{const.}$$

Starting from the finite form so obtained for the case of a single circle, a method which is essentially the same as the physical method of images may be applied to form a series, which, if convergent, will give the required function for the case of two circular boundaries.

The two circles might first, by a linear transformation, be replaced by two concentric circles, but the simplification of the algebraic work thereby introduced is not great; while the point at infinity being then a singular point for the functions involved, the result would lose something of its generality. Consider then the case in which the boundary consists of two non-intersecting circles

$$|\,z-z_1\,|=r_1\quad\text{and}\quad|\,z-z_2\,|=r_2.$$

By applying continually the previous result a series may be obtained of the form

$$w=\frac{A}{z-a}\begin{array}{l}+\dfrac{A_1}{z-a_1}+\dfrac{A_{12}}{z-a_{12}}+\ldots\\[2ex]+\dfrac{A_2}{z-a_2}+\dfrac{A_{21}}{z-a_{21}}+\ldots;\end{array}$$

where
$$A_1=-\left(\frac{r_1}{a'-z_1'}\right)^2A'e^{-2i\theta_1},$$

$$A_{12}=-\left(\frac{r_2}{a_1'-z_2'}\right)^2A_1'e^{-2i\theta_2};$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

and a_{12} is the inverse point of a_1 in the circle $|z-z_2|=r_2$, &c. Each term of this series corrects over one circle the error introduced over it by the preceding term, and hence, if the series is convergent, it clearly represents the function required.

The form of the series may be simplified by the following considerations.

Two consecutive inversions are equivalent to a linear substitution, and hence the quantities a, a_1, a_{12}, a_{21}, a_{121}, &c., in the denominators can be derived from a and a_1 (or from a and a_2) by transforming these by means of the direct or inverse powers of a linear substitution.

Suppose that
$$a_{12}=\frac{\alpha a+b}{ca+d};$$

any term $a_{121\ldots}$ or $a_{212\ldots}$ with an even number of suffixes can then be written in the form
$$\frac{a_n a+b_n}{c_n a+d_n},$$

where $\left(z,\ \dfrac{a_n z+b_n}{c_n z+d_n}\right)$ is a power of $\left(z,\ \dfrac{az+b}{cz+d}\right)$, and any term with an odd number of suffixes can be written in the form

$$\frac{a_n a_1+b_n}{c_n a_1+d_n}.$$

Also, since
$$a_{12}=z_2+\frac{r_2^2}{a_1'-z_2'},$$

therefore
$$da_{12}=-\left(\frac{r_2}{a_1'-z_2'}\right)^2da_1',$$

and $$da_1 = -\left(\frac{r_1}{a'-z_1'}\right)^2 da'.$$

Therefore $$A_{12} = -\left(\frac{r_2}{a_1'-z_2'}\right)^2 e^{-2i\theta_2} A_1'$$

$$= \left(\frac{r_2}{a_1'-z_2'}\right)^2 e^{-2i\theta_2} \left(\frac{r_1}{a-z_1}\right)^2 A e^{2i\theta_1}$$

$$= e^{2i(\theta_1-\theta_2)} A \frac{da_{12}}{da},$$

which result can be generalized at once.

The series therefore can be written in the form

$$w = A \sum_{-\infty}^{\infty} e^{2in(\theta_1-\theta_2)} \frac{1}{z-f_n(a)} \frac{df_n(a)}{da}$$

$$+ A_1 \sum_{-\infty}^{\infty} e^{2in(\theta_1-\theta_2)} \frac{1}{z-f_n(a_1)} \frac{df_n(a_1)}{da_1},$$

where f_n represents the result of repeating n times the linear substitution

$$\left(z, \frac{az+b}{cz+d}\right).$$

Now the substitution $$\left(z, \frac{az+b}{cz+d}\right)$$

has for its double points the limiting points of the coaxal system to which the two circles belong; while it may easily be verified that the multiplier is real and different from unity. It may therefore be written in the form

$$\frac{f(z)-g}{f(z)-h} = K\frac{z-g}{z-h},$$

and hence $$f_n(a) = \frac{(hK^n-g)\,a - hg\,(K^n-1)}{(K^n-1)\,a+h-gK^n};$$

therefore $$\frac{df_n(a)}{da} = \frac{(h-g)^2 K^n}{\left[(K^n-1)\,a+h-gK^n\right]^2}$$

$$= \frac{(h-g)^2}{\left[K^{\frac{1}{2}n}(a-g) - K^{-\frac{1}{2}n}(a-h)\right]^2}.$$

It follows that $\sum_{-\infty}^{\infty} \frac{df_n(a)}{da}$ is a uniformly convergent series, and therefore at once that the two infinite series in the expression for w are uniformly convergent and represent continuous functions of z.

If $\theta_1-\theta_2$ is commensurable with π, the expression for w can be put in a finite form by means of theta-functions, but not otherwise. Suppose, for instance, that

$$\theta_1-\theta_2=\frac{\pi}{p},$$

where p is an integer; then the terms in w which contain $Ae^{(2is\pi)/p}$ $(s<p)$ as a coefficient are

$$\sum_{-\infty}^{+\infty}\frac{1}{z-f_{s+np}(a)}\,\frac{df_{s+np}(a)}{da}.$$

Now $$\frac{1}{z-f_{s+np}(a)}\,\frac{df_{s+np}(a)}{da}+\frac{1}{z-f_{s-np}(a)}\,\frac{df_{s-np}(a)}{da}$$

$$=-\frac{\partial}{\partial a}\log\left[z-\frac{(hK^{np}-g)\,f_s(a)-hg\,(K^{np}-1)}{(K^{np}-1)\,f_s(a)+h-gK^{np}}\right]$$

$$\times\left[z-\frac{(hK^{-np}-g)\,f_s(a)-hg\,(K^{-np}-1)}{(K^{-np}-1)\,f_s(a)+h-gK^{-np}}\right]$$

$$=-\frac{\partial}{\partial a}\log\frac{(z-h)\left[f_s(a)-g\right]-(z-g)\left[f_s(a)-h\right]K^{-np}}{f_s(a)-g-\left[f_s(a)-h\right]K^{-np}}$$

$$\times\frac{(z-g)\left[f_s(a)-h\right]-(z-h)\left[f_s(a)-g\right]K^{-np}}{f_s(a)-h-\left[f_s(a)-g\right]K^{-np}}$$

$$=-\frac{\partial}{\partial a}\log\frac{1-\dfrac{z-g}{z-h}\,\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}}{1-\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}}\;\frac{1-\dfrac{z-h}{z-g}\,\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}}{1-\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}}.$$

The terms in question can then be written

$$-\frac{\partial}{\partial a}\log\frac{\left(1-\dfrac{z-g}{z-h}\,\dfrac{f_s(a)-h}{f_s(a)-g}\right)\displaystyle\prod_1^{\infty}\left(1-\dfrac{z-g}{z-h}\,\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}\right)\times\left(1-\dfrac{z-h}{z-g}\,\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}\right)}{\left(1-\dfrac{f_s(a)-h}{f_s(a)-g}\right)\displaystyle\prod_1^{\infty}\left(1-\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}\right)\left(1-\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}\right)},$$

where finally the denominator can be omitted as a mere constant.

Omitting a further constant, this is the same as

$$-\frac{\partial}{\partial_u}\log\theta_1(x_s, q),$$

where $$q = K^{-ip},$$

and $$x_s = \frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{f_s(a)-h}{f_s(a)-g} = \frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a-h}{a-g} + \frac{is}{p}\log q.$$

Now $$\frac{\partial x_s}{\partial a} = \frac{1}{2i}\,\frac{h-g}{(a-h)(a-g)},$$

and the part of w which has A for a coefficient can therefore be written in the form

$$\frac{iA}{2}\,\frac{h-g}{(a-h)(a-g)}\sum_{s=0}^{s=p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(x_s)}{\theta_1(x_s)}.$$

The quantity $\dfrac{h-g}{(a_1-h)(a_1-g)}$, which occurs as a factor in the other half of w, may easily be put in the form

$$\left(\frac{a'-z_1'}{r_1}\right)^2\,\frac{h'-g'}{(a'-h')(a'-g')},$$

so that, if y_s is written for the result of putting a_1 for a in x_s, the second part of w becomes

$$-\frac{iA'}{2}\,\frac{h'-g'}{(a'-h')(a'-g')}\,e^{-2i\vartheta_1}\sum_{s=0}^{s=p-} e^{(2is\pi)/p}\,\frac{\theta_1'(y_s)}{\theta_1(y_s)}.$$

Finally, simplifying the constants by writing

$$\frac{iA}{2}\,\frac{h-g}{(a-h)(a-g)}\,e^{i\vartheta_1} = B,$$

$$e^{i\vartheta_1}w = B\sum_0^{p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(x_s)}{\theta_1(x_s)} + B'\sum_0^{p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(y_s)}{\theta_1(y_s)}.$$

The function so obtained represents the space outside two non-intersecting circles conformably on an infinite plane in which two finite non-intersecting straight lines inclined at an angle π/p are to be regarded as boundaries.

If $\theta_1 = \theta_2 = 0$, so that $p = 1$,

$$w = B\,\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a-h}{a-g}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a-h}{a-g}\right)} + B'\,\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a_1-h}{a_1-g}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a_1-h}{a_1-g}\right)}$$

is a function having a single infinity at a in the space outside the two circles, and constant imaginary part at their circumferences.

If, further, $a = \infty$, and therefore

$$a_1 = z_1,$$

$$w = B\,\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\right)} + B'\,\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h} - \frac{1}{2i}\log\frac{z_1-g}{z_1-h}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h} - \frac{1}{2i}\log\frac{z_1-g}{z_1-h}\right)},$$

where the modulus of the theta-functions is $K^{-\frac{1}{2}}$, gives the uniform streaming motion of fluid in any direction past two right circular cylinders. This particular case of functions of the kind considered has already been dealt with by Mr. Hicks and Mr. Greenhill in Vols. XIV. and XVI. of the *Quarterly Journal of Mathematics*.

In this case the original expression for w is

$$w = A\Sigma\frac{1}{z-f_n(a)}\,\frac{df_n(a)}{da} + A_1\Sigma\frac{1}{z-f_n(a_1)}\,\frac{df_n(a_1)}{da_1};$$

therefore

$$\frac{\partial w}{\partial z} = -A\Sigma\frac{1}{\left[z-f_n(a)\right]^2}\,\frac{df_n(a)}{da} - A_1\Sigma\frac{1}{\left[z-f_n(a_1)\right]^2}\,\frac{df_n(a_1)}{da_1}.$$

Now, if $\qquad f_n(a) = \dfrac{a_n a + b_n}{c_n a + d_n}, \quad \left[a_n d_n - b_n c_n = 1\right],$

$$\frac{1}{\left[z-f_n(a)\right]^2}\,\frac{df_n(a)}{da} = \frac{1}{\left[z(c_n a + d_n) - c_n a - b_n\right]^2}$$

$$= \frac{(-c_n z + a_n)^{-2}}{\left(a - \dfrac{d_n z - b_n}{-c_n z + a_n}\right)^2}$$

$$= \frac{1}{\left[a - f_{-n}(z)\right]^2}\,\frac{df_{-n}(z)}{dz},$$

or
$$\frac{\partial w}{\partial z} = -\Sigma\left[\frac{A}{\left[a-f_n(z)\right]^2}+\frac{A_1}{\left[a_1-f_n(z)\right]^2}\right]\frac{df_n(z)}{dz}$$

$$= -\Sigma\left[\frac{A}{\left[a-f_n(z)\right]^2}+\frac{A_1}{\left[a_1-f_n(z)\right]^2}\right](c_n z+d_n)^{-2}.$$

The form of this function is precisely that of those that M. Poincaré calls theta-fuchsian, except that the factor $c_n z+d_n$ occurs in the power -2, while the general form of a theta-fuchsian function as considered by M. Poincaré, is

$$\sum_n \phi\left(\frac{a_n z+b_n}{c_n z+d_n}\right)(c_n z+d_n)^{-2m},$$

where m is not less than 2.

The group of substitutions to which the above $\frac{\partial w}{\partial z}$ belongs, being derived from a single fundamental substitution, is the simplest possible, and is only incidentally referred to by M. Poincaré in his memoir (*Acta Mathematica*, Vol. I.).

The previous investigation shows that for the discontinuous infinite group derived from a single fundamental substitution (which must be hyperbolic or loxodromic to give a discontinuous group),

$$\Sigma\,(c_n z+d_n)^{-2}$$

is a convergent series, and that therefore theta-fuchsian functions of the form

$$\sum_n \phi\left(\frac{a_n z+b_n}{c_n z+d_n}\right)(c_n z+d_n)^{-}$$

do really exist.

Now, if
$$\frac{\partial w}{\partial z} = f(z),$$

then, from the fundamental properties of these functions,

$$f\left(\frac{az+b}{cz+d}\right) = f(z)\,(cz+d)^2;$$

and therefore
$$f\left(\frac{az+b}{cz+d}\right)d\,\frac{az+b}{cz+d} = f(z)\,dz;$$

or, if
$$z' = \frac{az+b}{cz+d},$$

$$\int^{z'} f(z')\,dz' = \int^{z} f(z)\,dz,$$

so that the variable part of w must remain unchanged by the substitution
$$\left(z, \frac{az+b}{cz+d}\right),$$

or, in other words, by any substitution of the group.

The function w therefore has its characteristic properties, not only in the original region R, but in any of the regions into which this may be transformed by the substitutions of the group.

These latter results can be derived directly from the expression of w in theta-functions; but with a view to the consideration of the case when the boundary consists of three or more circles, in which the group of substitutions involved is derived from more than one fundamental substitution, and the functions can therefore not be expressed in terms of theta-functions of one variable, it appears desirable to show how in this simplest case the idea of a group of substitutions and of functions which are unchanged by it is involved.

Before going on to consider the functions for streaming motion and the analogous functions considered here for the case of three or more circular boundaries, it will be necessary to investigate the convergence of series analogous to the one just dealt with, but arising from a more extended group of substitutions. This I shall do in another paper, which I hope to have the honour of laying before the Society.

Relations between the Divisors of the First n Numbers.

By J. W. L. GLAISHER, Sc.D., F.R.S.

[*Read May 14th,* 1891.]

Introduction. § 1.

1. The present paper contains various generalizations of the recurring formula

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+9\sigma(n-10)-\&c. = 0,^*$$

in which $\sigma(n)$ denotes the sum of the divisors of the number n, and $\sigma(0)$, when it occurs, is to have the value $\frac{1}{3}n$.

It will be seen that the theorems in their most general form relate to the actual divisors themselves, and not necessarily to their sums or other numbers obtained from them by any method of combination. We may, however, deduce various theorems of the same kind as the one quoted above by so combining them; and numerous examples of results obtained in this manner occur in the paper. These formulæ relate not only to the sums of the divisors, but also to the sums of their m^{th} powers, m being any uneven number.†

Notation. § 2.

2. Let $G_n\{\phi(d), \psi(d), \chi(d), \ldots\}$ denote the group of numbers

$$\begin{array}{l} \phi(d_1),\ \phi(d_2),\ \phi(d_3),\ \ldots,\ \phi(d_f), \\ \psi(d_1),\ \psi(d_2),\ \psi(d_3),\ \ldots,\ \psi(d_f), \\ \chi(d_1),\ \chi(d_2),\ \chi(d_3),\ \ldots,\ \chi(d_f), \\ \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \end{array}$$

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* *Quarterly Journal of Mathematics,* Vol. xix., p. 220, and *Proc. Camb. Phil. Soc.,* Vol. v., p. 109.

† The general theorems in §§ 3, 11, 29, 46, 58 were obtained in December, 1887, and the first of the theorems relating to the actual divisors (§ 3), with the derived numerical theorem (§ 11), was brought before the Society at its meeting on January 12th, 1888 ("A Theorem connecting the Divisors of a certain Series of Numbers," Vol. xix., p. 143); but no paper was prepared for publication at that time. I did not return to the subject till December, 1890; since then I have worked out the details and made considerable additions, forming the present paper. A few short notes containing investigations arising out of the subject, and which have been published elsewhere, are referred to in the notes to §§ 21, 43, 53.

where $d_1, d_2, d_3, \ldots, d_r$ are all the divisors of the number n (which, it will be observed, occurs as the suffix of the letter G). The numbers 1 and n are to be included among the divisors; and it is supposed that

$$-G_n\{\phi(d), \psi(d), \chi(d), \ldots\},$$

is the same as $G_n\{-\phi(d), -\psi(d), -\chi(d), \ldots\}$.

General Theorem relating to the Actual Divisors of the Numbers $n, n-1, n-3$, *&c.* §§ 3–8.

3. Using the above notation, the general theorem may be expressed as follows:—The numbers given by the formula

$$G_n(d) - G_{n-1}(d, d\pm 1) + G_{n-3}(d, d\pm 1, d\pm 2)$$
$$- G_{n-6}(d, d\pm 1, d\pm 2, d\pm 3) + \&c.$$

all cancel each other, if n is not a triangular number, and reduce to

$$(-1)^{r-1} (\text{one } 1, \text{ two } 2\text{'s}, \text{ three } 3\text{'s}, \ldots, r\ r\text{'s}),$$

if n is the r^{th} triangular number $\frac{1}{2}r(r+1)$.

4. For example, putting $n = 5$, which is not a triangular number, the theorem asserts that the numbers given by the formula

$$G_5(d) - G_4(d, d\pm 1) + G_2(d, d\pm 1, d\pm 2) [illegible]$$

all cancel each other.

Writing the numbers [illegible] in lines above the divisors [illegible] and the numbers [illegible] in lines below, so that the central line consists of the divisors of the numbers 5, 4, 2, [illegible] the numbers [illegible] the formula becomes

[illegible]

or, changing the signs of the numbers occurring in the groups to which the negative sign is prefixed,

$$\begin{array}{rrrr}
 & & & -4,\ -6, \\
 & & 3,\ 4,\ 5,\ 8, & -3,\ -5, \\
 & -2,\ -3,\ -5,\ -9, & 2,\ 3,\ 4,\ 7, & -2,\ -4, \\
1,\ 3,\ 9, & -1,\ -2,\ -4,\ -8, & 1,\ 2,\ 3,\ 6, & -1,\ -3, \\
 & 0,\ -1,\ -3,\ -7, & 0,\ 1,\ 2,\ 5, & 0,\ -2, \\
 & & -1;\ 0,\ 1,\ 4, & 1,\ -1, \\
 & & & 2,\ \ 0,
\end{array}$$

and it is easy to verify that these numbers exactly cancel each other, *i.e.*, there are five 1's and five −1's, four 2's and four −2's, four 3's and four −3's, three 4's and three −4's, two 5's and two −5's, one 6 and one −6, one 7 and one −7, one 8 andone −8, one 9 and one −9. The zeros are retained for the sake of regularity in writing the numbers, but no account is to be taken of them.

As a second example, putting $n = 10$, which is the fourth triangular number, so that $g = 4$, the theorem asserts that the numbers given by the formula

$$G_{10}(d) - G_9(d,\ d \pm 1) + G_7(d,\ d \pm 1,\ d \pm 2) - G_4(d,\ d \pm 1,\ d \pm 2,\ d \pm 3)$$

all cancel each other, excepting only

$$-1,\ -2,\ -2,\ -3,\ -3,\ -3,\ -4,\ -4,\ -4,\ -4.$$

The divisors of 10, 9, 7, 4 are 1, 2, 5, 10; 1, 3, 9; 1, 7; 1, 2, 4 respectively, so that the numbers given by the formula are

$$\{1,\ 2,\ 5,\ 10\} - \left\{\begin{matrix} 2,\ 4,\ 10 \\ 1,\ 3,\ \ 9 \\ 0,\ 2,\ \ 8 \end{matrix}\right\} + \left\{\begin{matrix} 3,\ 9 \\ 2,\ 8 \\ 1,\ 7 \\ 0,\ 6 \\ -1,\ 5 \end{matrix}\right\} - \left\{\begin{matrix} 4, & 5,\ 7 \\ 3, & 4,\ 6 \\ 2, & 3,\ 5 \\ 1, & 2,\ 4 \\ 0, & 1,\ 3 \\ -1. & 0,\ 2 \\ -2, & -1,\ 1 \end{matrix}\right\};$$

that is

$$\begin{array}{rrrrrrrrrrrr}
 & & & & & & & & & -4, & -5, & -7, \\
 & & & & & & & 3, & 9, & -3, & -4, & -6, \\
 & & & & -2, & -4, & -10, & 2, & 8, & -2, & -3, & -5, \\
1, & 2, & 5, & 10, & -1, & -3, & -9, & 1, & 7, & -1, & -2, & -4, \\
 & & & & 0, & -2, & -8, & 0, & 6, & 0, & -1, & -3, \\
 & & & & & & & -1, & 5, & 1, & 0, & -2, \\
 & & & & & & & & & 2, & 1, & -1,
\end{array}$$

all of which cancel each other, excepting only one -1, two -2's, three -3's, and four -4's.

5. It is evident that we may express the theorem also in the form:—If $\alpha, \beta, \ldots$, be the divisors of n; $\alpha_1, \beta_1, \ldots$, those of $n-1$; $\alpha_2, \beta_2, \ldots$, those of $n-3$; $\alpha_3, \beta_3, \ldots$ of $n-6$, and so on, then the numbers

$$\{\alpha, \beta, \ldots\} - \left\{\begin{matrix} \alpha_1+1, & \beta_1+1, & \ldots \\ \alpha_1, & \beta_1, & \ldots \\ \alpha_1-1, & \beta_1-1, & \ldots \end{matrix}\right\} + \left\{\begin{matrix} \alpha_2+2, & \beta_2+2, & \ldots \\ \alpha_2+1, & \beta_2+1, & \ldots \\ \alpha_2, & \beta_2, & \ldots \\ \alpha_2-1, & \beta_2-1, & \ldots \\ \alpha_2-2, & \beta_2-2, & \ldots \end{matrix}\right\} - \&c.$$

cancel each other in all cases if we make a certain convention with respect to the group having the central line $\alpha_g, \beta_g, \ldots$, *i.e.*, if we suppose that this group, which can only occur when $n = \frac{1}{2}g\,(g+1)$, and which would be written

$$\left\{\begin{matrix} \alpha_g+g, & \beta_g+g, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_g+1, & \beta_g+1, & \ldots \\ \alpha_g, & \beta_g, & \ldots \\ \alpha_g-1, & \beta_g-1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_g-g, & \beta_g-g, & \ldots \end{matrix}\right\},$$

is to be conventionally replaced by the group of numbers

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..., g, g, g ... (g times).

6. To obtain the system of numbers given by the formula, we first write down the divisors of n, $n-1$, $n-3$, $n-6$, &c. (forming the

central line in the above scheme). Commencing with the divisors of $n-1$, we next write down the numbers derived from them by increasing and diminishing each divisor by 1 (forming the lines next above and next below the central line). Commencing with the divisors of $n-3$, we next write down the numbers derived from the divisors in the central line by increasing and diminishing them by 2 (forming the next lines above and below); and so on. We then change the signs of the divisors of $n-1$, $n-6$, $n-15$, &c., and also of all the numbers derived from them. The theorem asserts that in the system of numbers so formed (if we adopt the above convention with respect to the group depending upon the divisors of $n-n$, when it occurs) every number appears an equal number of times with the positive and with the negative sign, so that all the numbers in the system cancel each other.

7. Retaining the G-notation, the theorem may be conveniently stated in the form :—The system of numbers given by the formula

$$G_n(d) - G_{n-1}\begin{Bmatrix} d+1 \\ d \\ d-1 \end{Bmatrix} + G_{n-1-2}\begin{Bmatrix} d+2 \\ d+1 \\ d \\ d-1 \\ d-2 \end{Bmatrix} - G_{n-1-2-3}\begin{Bmatrix} d+3 \\ d+2 \\ d+1 \\ d \\ d-1 \\ d-2 \\ d-3 \end{Bmatrix} + \&c.,$$

all cancel each other, subject to the convention that, if

$$n = 1+2+3+\dots+g,$$

then the last term

$$G_{n-1-2-\dots-g}\begin{Bmatrix} d+g \\ \dots \\ d+1 \\ d \\ d-1 \\ \dots \\ d-g \end{Bmatrix}$$

is to be replaced by the group of numbers

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ \dots\ g,\ g,\ g,\ \dots\ (g \text{ times}).$$

8. The theorem seems to be a very curious one, relating as it does to the mutual destruction of certain numbers depending upon the divisors of numbers separated from each other by fixed intervals. It assigns all the divisors of n when we know the divisors of the numbers which are inferior to n and separated from it by the intervals 1, 3, 6, &c. It thus effects the complete resolution of any number n into its factors, or in other words, lays bare the structure of the number n as regards its real divisors, when the structure of the numbers $n-1$, $n-3$, $n-6$, &c., is known. The recurring formula in § 1 suffices to determine whether n is prime or not, when we know the sums of the divisors of $n-1$, $n-3$, $n-6$, &c., for in that case $\sigma(n)$ is equal to $n+1$; but the general theorem goes further, and gives all the divisors of n by means of those of $n-1$, $n-3$, &c.

Numerical Theorems relating to the Sums of Powers of the Divisors of n, $n-1$, $n-3$, *&c.* §§ 9–13.

9. Since the actual numbers cancel each other in the general theorem, we may replace them by any function of themselves, so long as the function is the same for all and changes sign with the argument. We may, further, combine all the functions in each group by addition, and thus derive from the general theorem a numerical equation connecting together functions of the divisors of the numbers n, $n-1$, $n-3$, &c.

Let $$\Sigma_n \{\phi(d),\ \psi(d),\ \chi(d),\ \ldots\}$$

denote the sum

$$\begin{aligned}&\phi(d_1)+\phi(d_2)+\ldots+\phi(d_\nu)\\ &+\psi(d_1)+\psi(d_2)+\ldots+\psi(d_\nu)\\ &+\chi(d_1)+\chi(d_2)+\ldots+\chi(d_\nu)+\ldots,\end{aligned}$$

where $d_1, d_2, \ldots d_\nu$ are, as in § 2, all the divisors of n.

By replacing $G(d, d\pm 1, \ldots)$ by $\Sigma\{\phi(d), \phi(d\pm 1), \ldots\}$, ϕ being any uneven function, we find that

$$\begin{aligned}\Sigma_n\phi(d)-\Sigma_{n-1}\{\phi(d)+\phi(d\pm 1)\}+\Sigma_{n-3}\{\phi(d)+\phi(d\pm 1)+\phi(d\pm 2)\}\\ -\Sigma_{n-6}\{\phi(d)+\phi(d\pm 1)+\phi(d\pm 2)+\phi(d\pm 3)\}+\&c.\end{aligned}$$

$= 0$, if n is not a triangular number.

and $$= (-1)^{g-1}\{\phi(1)+2\phi(2)+3\phi(3)+\ldots+g\phi(g)\},$$

if $$n = 1+2+3+\ldots+g = \tfrac{1}{2}g(g+1).$$

10. As a particular case, putting

$$\phi(d) = d,$$

the equation becomes

$$\Sigma_n d-\Sigma_{n-1}(3d)+\Sigma_{n-3}(5d)-\Sigma_{n-6}(7d)+\Sigma_{n-10}(9d)-\&c.$$
$$= 0 \text{ or } (-1)^{g-1}(1^2+2^2+3^2+\ldots+g^2),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g.$$

Putting $$\phi(d) = d^3,$$

it becomes

$$\Sigma_n(d^3)-\Sigma_{n-1}(3d^3+6d)+\Sigma_{n-3}\{5d^3+6(1^2+2^2)d\}$$
$$-\Sigma_{n-6}\{7d^3+6(1^2+2^2+3^2)d\}+\Sigma_{n-10}\{9d^3+6(1^2+2^2+3^2+4^2)d\}-\&c.$$
$$= 0, \text{ or } (-1)^{g-1}(1^4+2^4+3^4+\ldots+g^4),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g.$$

Putting $\phi(d) = d^5$,

$$\Sigma_n(d^5)-\Sigma_{n-1}(3d^5+20d^3+10d)+\Sigma_{n-3}\{5d^5+20(1^2+2^2)d^3+10(1^4+2^4)l\{$$
$$-\Sigma_{n-6}\{7d^5+20(1^2+2^2+3^2)d^3+10(1^4+2^4+3^4)d\}+\&c.$$
$$= 0, \text{ or } (-1)^{g-1}(1^6+2^6+3^6+\ldots+g^6),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g;$$

and so on.

11. Denoting by $\sigma_r(n)$ the sum of the r^{th} powers of the divisors of n, we may write these equations in the following form, the additional term in square brackets coming into existence only in the case when n is a triangular number $\frac{1}{2}g(g+1)$.

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+9\sigma(n-10)-\&c.$$
$$= \left[(-1)^{g-1}(1^2+2^2+3^2+\ldots+g^2)\right],$$

$$\sigma_3(n)-3\sigma_3(n-1)+5\sigma_3(n-3)-7\sigma_3(n-6)+9\sigma_3(n-10)-\&c.$$

$$= \quad 6\left\{\sigma\ (n)-(1^2+2^2)\,\sigma\ (n-3)+(1^2+2^2+3^2)\,\sigma\ (n-6)-\&c.\right\}$$

$$+\left[(-1)^{\rho-1}(1^4+2^4+3^4+\ldots+g^4)\right],$$

$$\sigma_5(n)-3\sigma_5(n-1)+5\sigma_5(n-3)-7\sigma_5(n-6)+9\sigma_5(n-10)-\&c.$$

$$= \quad 20\left\{\sigma_3(n)-(1^2+2^2)\,\sigma_3(n-3)+(1^2+2^2+3^2)\,\sigma_3(n-6)-\&c.\right\}$$

$$+\ 10\left\{\sigma\ (n)-(1^4+2^4)\,\sigma\ (n-3)+(1^4+2^4+3^4)\,\sigma\ (n-6)-\&c.\right\}$$

$$+\left[(-1)^{\rho-1}(1^6+2^6+3^6+\ldots+g^6)\right],$$

$$\sigma_7(n)-3\sigma_7(n-1)+5\sigma_7(n-3)-7\sigma_7(n-6)+9\sigma_7(n-10)-\&c.$$

$$= \quad 42\left\{\sigma_5(n-1)-(1^2+2^2)\,\sigma_5(n-3)+(1^2+2^2+3^2)\,\sigma_5(n-6)-\&c.\right\}$$

$$+\ 70\left\{\sigma_3(n-1)-(1^4+2^4)\,\sigma_3(n-3)+(1^4+2^4+3^4)\,\sigma_3(n-6)-\&c.\right\}$$

$$+\ 14\left\{\sigma\ (n-1)-(1^6+2^6)\,\sigma\ (n-3)+(1^6+2^6+3^6)\,\sigma\ (n-6)-\&c.\right\}$$

$$+\left[(-1)^{\rho-1}(1^8+2^8+3^8+\ldots+g^8)\right];$$

and, in general, m being any uneven number,

$$\sigma_m(n)-3\sigma_m(n-1)+5\sigma_m(n-3)-7\sigma_m(n-6)+9\sigma_m(n-10)-\&c.$$

$$= \quad 2\,\frac{m(m-1)}{2!}\left\{\sigma_{m-2}(n-1)-(1^2+2^2)\,\sigma_{m-2}(n-3)\right.$$

$$\left.+(1^2+2^2+3^2)\,\sigma_{m-2}(n-6)-\&c.\right\}$$

$$+2\,\frac{m(m-1)(m-2)(m-3)}{4!}\left\{\sigma_{m-4}(n-1)-(1^4+2^4)\,\sigma_{m-4}(n-3)\right.$$

$$\left.+(1^4+2^4+3^4)\,\sigma_{m-4}(n-6)-\&c.\right\}$$

... ...

$$+2m\left\{\sigma(n-1)-(1^{m-1}+2^{m-1})\,\sigma(n-3)\right.$$

$$\left.+(1^{m-1}+2^{m-1}+3^{m-1})\,\sigma(n-6)-\&c.\right\}$$

$$+\left[(-1)^{\rho-1}(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1})\right].$$

12. This general theorem expresses the sum of the mth powers (m being uneven) of the divisors of n in terms of the sums of the mth and lower powers of the divisors of $n-1$, $n-3$, $n-6$, &c. The form of the right hand member of the equation is curious; all the coefficients, and even the additional term which occurs when n is equal to $\frac{1}{2}g(g+1)$, being sums of even powers of the natural

numbers. The nature of the theorem is best seen by taking a particular case. Putting therefore $n=9$ (which is not a triangular number, so that the additional term does not occur), the theorem gives

$$\begin{aligned}
&1^m+3^m+9^m-3(1^m+2^m+4^m+8^m)+5(1^m+2^m+3^m+6^m)-7(1^m+3^m)\\
&= \quad 2\frac{m(m-1)}{2!}\{1^{m-2}+2^{m-2}+4^{m-2}+8^{m-2}\\
&\qquad\qquad -(1^2+2^2)(1^{m-2}+2^{m-2}+3^{m-2}+6^{m-2})\\
&\qquad\qquad +(1^2+2^2+3^2)(1^{m-2}+3^{m-2})\}\\
&\quad +2\frac{m(m-1)(m-2)(m-3)}{4!}\{1^{m-4}+2^{m-4}+4^{m-4}+8^{m-4}\\
&\qquad\qquad -(1^4+2^4)(1^{m-4}+2^{m-4}+3^{m-4}+6^{m-4})\\
&\qquad\qquad +(1^4+2^4+3^4)(1^{m-4}+3^{m-4})\}\\
&\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\\
&\quad +2m\{1+2+4+8-(1^{m-1}+2^{m-1})(1+2+3+6)\\
&\qquad\qquad +(1^{m-1}+2^{m-1}+3^{m-1})(1+3)\}.
\end{aligned}$$

Thus the theorem connects together the sums of uneven powers of the divisors of numbers separated from each other by the intervals 1, 2, 3, ..., and the sums of even powers of the natural numbers.

13. It will be observed that in the general theorem in § 11, we may dispense with the additional term if we assign to $\sigma_m(0)$ the conventional meaning

$$\frac{1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1}}{2g+1},$$

and put $\sigma_{m-2}(0)$, $\sigma_{m-4}(0)$, ... $\sigma(0)$ equal to zero. The coefficient of $\sigma_m(0)$ is $2g+1$, and we are therefore to replace $(2g+1)\,\sigma_m(0)$ by $1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1}$, the value of g being derivable from the coefficient of $\sigma_m(0)$ without the necessity of having recourse to the formula $n=\frac{1}{2}g(g+1)$.*

Another convention by which the additional term in the general theorem may be conveniently represented is given in §§ 20–22.

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* The number g is also equal to the number of terms in the series on the left-hand side of the equation, the term $(2g+1)\,\sigma_m(0)$ not being counted.

The particular cases of $m = 1$, 3, and 5. §§ 14–17.

14. In examining the first few particular cases of the general theorem in § 11, it seems worth while in the first place to take a numerical example of each theorem, both in the case when n is not, and when n is, a triangular number. Putting $n = 9$, which is not a triangular number, the theorems give

$$1+3+9-3\,(1+2+4+8)+5\,(1+2+3+6)-7\,(1+3) = 0.$$

$$\begin{aligned} 1^3+3^3+9^3-3\,(1^3+2^3+4^3+8^3)+5\,(1^3+2^3+3^3+6^3)-7\,(1^3+3^3) \\ = 6\,\{1+2+4+8-5\,(1+2+3+6)+14\,(1+3)\}, \end{aligned}$$

$$\begin{aligned} 1^5+3^5+9^5-3\,(1^5+2^5+4^5+8^5)+5\,(1^5+2^5+3^5+6^5)-7\,(1^5+3^5) \\ = 20\,\{1^3+2^3+4^3+8^3-5\,(1^3+2^3+3^3+6^3)+14\,(1^3+3^3)\} \\ +10\,\{1+2+4+8-17\,(1+2+3+6)+98\,(1+3)\}; \end{aligned}$$

and, putting $n=10$, which $= 1+2+3+4$, so that $g = 4$, the theorems give

$$\begin{aligned} 1+2+5+10-3\,(1+3+9)+5\,(1+7)-7\,(1+2+4) \\ = -\,(1^2+2^2+3^2+4^2), \end{aligned}$$

$$\begin{aligned} 1^3+2^3+5^3+10^3-3\,(1^3+3^3+9^3)+5\,(1^3+7^3)-7\,(1^3+2^3+4^3) \\ = 6\,\{1+3+9-5\,(1+7)+14\,(1+2+4)\} \\ -\,(1^4+2^4+3^4+4^4), \end{aligned}$$

$$\begin{aligned} 1^5+2^5+5^5+10^5-3\,(1^5+3^5+9^5)+5\,(1^5+7^5)-7\,(1^5+2^5+4^5) \\ = 20\,\{1^3+3^3+9^3-5\,(1^3+7^3)+14\,(1^3+2^3+4^3)\} \\ +10\,\{1+3+9-17\,(1+7)+98\,(1+2+4)\} \\ -\,(1^6+2^6+3^6+4^6). \end{aligned}$$

The correctness of these equalities is easily verified.

15. In spite of the fact that some of the elegance and regularity of the formulæ is lost by summing the powers of the natural numbers which form the coefficients, it is still interesting to exhibit the series in a form in which the coefficients of the general terms are expressed algebraically.

Since $\quad 1^2+2^2+3^2+\ldots+i^2 = \tfrac{1}{6}i\,(i+1)(2i+1),$

the theorem in the case of $m=1$ may be written

$$\Sigma\,(-1)^i\,(2i+1)\,\sigma\left\{n-\tfrac{1}{2}i\,(i+1)\right\}=\left[(-1)^{g-1}\tfrac{1}{6}g\,(g+1)(2g+1)\right],$$

the summation extending from $i=0$ to $i=h$, where $\frac{1}{2}h\,(h+1)$ is the triangular number next inferior to n.

When $n=\frac{1}{2}g\,(g+1)$, we obtain the term $(-1)^g\,(2g+1)\,\sigma\,(0)$ by continuing the series one term further. We may therefore dispense with the additional term if we extend the summation from $i=0$ to $i=k$, where $\frac{1}{2}k\,(k+1)$ is the triangular number nearest to, and not exceeding, n (*i.e.*, so that $\frac{1}{2}k\,(k+1)$ is the triangular number next inferior to n, if n is not a triangular number, and is equal to n if n is a triangular number), and assign the conventional value $\frac{1}{3}n$ to $\sigma\,(0)$, when it occurs. This is the form in which the theorem was quoted in § 1.

16. Since

$$1^4+2^4+3^4+\ldots+i^4=\tfrac{1}{30}i\,(i+1)(2i+1)(3i^2+3i-1),$$

we may write the theorem, in the case of $m=3$, in the form

$$\begin{aligned}&\Sigma\,(-1)^i\,(2i+1)\,\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}\\&=\Sigma\,(-1)^{i-1}\,i\,(i+1)(2i+1)\,\sigma\left\{n-\tfrac{1}{2}i\,(i+1)\right\}\\&\qquad+\left[(-1)^{g-1}\tfrac{1}{30}g\,(g+1)(2g+1)(3g^2+3g-1)\right],\end{aligned}$$

the summation extending from $i=0$ to $i=h$, where $\frac{1}{2}h\,(h+1)$ is the triangular number next inferior to n.

The additional term, which only occurs when $n=\frac{1}{2}g\,(g+1)$, may be written in the form

$$(-1)^{g-1}\,\tfrac{1}{15}\,(2g+1)\,n\,(6n-1)\,;$$

we may therefore dispense with it if we extend the summations so as to include the argument zero, and define $\sigma\,(0)$ and $\sigma_3\,(0)$ by any of the three following pairs of equations:—

$$\text{(i.)}\ \ \sigma\,(0)=0,\qquad \sigma_3\,(0)=\tfrac{2}{5}n^2-\tfrac{1}{15}n,$$

$$\text{(ii.)}\ \ \sigma\,(0)=\tfrac{1}{5}n-\tfrac{1}{30},\qquad \sigma_3\,(0)=0,$$

$$\text{(iii.)}\ \ \sigma\,(0)=\tfrac{1}{5}n,\qquad \sigma_3\,(0)=-\tfrac{1}{15}n.$$

Adopting any one of these three pairs of simultaneous values of $\sigma\,(0)$ and $\sigma_3\,(0)$, we may write the theorem

$$\begin{aligned}&\Sigma\,(-1)^i\,(2i+1)\,\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}\\&\qquad+\Sigma\,(-1)^i\,i\,(i+1)(2i+1)\,\sigma\left\{n-\tfrac{1}{2}i\,(i+1)\right\}=0,\end{aligned}$$

the summation now extending from $i = 0$ to $i = k$, where $\frac{1}{2}k(k+1)$ is the triangular number which is nearest to, and does not exceed, n.

For example, putting $n = 10$, and adopting the last of the three conventions, the theorem gives

$$\sigma_3(10)-3\sigma_3(9)+5\sigma_3(7)-7\sigma_3(4)+9\sigma_3(0)$$
$$= 6\sigma(9)-30\sigma(7)+84\sigma(4)-180\sigma(0),$$

where $$\sigma(0) = 2, \quad \sigma_3(0) = -\tfrac{2}{3}.$$

17. Since

$$1^6+2^6+3^6+\ldots+i^6 = \tfrac{1}{42}i(i+1)(2i+1)\{3i^4+6i^3-3i+1\},$$

the theorem, in the case of $m = 5$, may be written

$$\Sigma(-1)^i(2i+1)\,\sigma_5\{n-\tfrac{1}{2}i(i+1)\}$$
$$= \tfrac{10}{3}\Sigma(-1)^{i-1}i(i+1)(2i+1)\,\sigma_3\{n-\tfrac{1}{2}i(i+1)\}$$
$$+\tfrac{1}{3}\Sigma(-1)^{i-1}i(i+1)(2i+1)(3i^2+3i-1)\,\sigma\{n-\tfrac{1}{2}i(i+1)\}$$
$$+\left[(-1)^{g-1}\tfrac{1}{42}g(g+1)(2g+1)(3g^4+6g^3-3g+1)\right],$$

the limits of summation being $i = 0$ and $i = h$, where h has the same meaning as in the preceding section.

The additional term may be written in the form

$$(-1)^{g-1}\tfrac{1}{21}(2g+1)\,n\,(12n^2-6n+1);$$

we may therefore dispense with it if we extend the summations so as to include the argument zero, and define $\sigma(0)$, $\sigma_3(0)$, $\sigma_5(0)$ by any of the three sets of equations:

$$\text{(i.)}\ \sigma(0) = 0, \quad \sigma_3(0) = 0, \quad \sigma_5(0) = \tfrac{4}{7}n^3-\tfrac{2}{7}n^2+\tfrac{1}{21}n,$$
$$\text{(ii.)}\ \sigma(0) = 0, \quad \sigma_3(0) = \tfrac{3}{35}n^2-\tfrac{3}{70}n+\tfrac{1}{140}, \quad \sigma_5(0) = 0,$$
$$\text{(iii.)}\ \sigma(0) = \tfrac{1}{7}n, \quad \sigma_3(0) = -\tfrac{1}{35}n, \quad \sigma_5(0) = \tfrac{1}{21}n.$$

Assigning to $\sigma(0)$, $\sigma_3(0)$, $\sigma_5(0)$ any one of these three sets of values, we may write the theorem in the form

$$\Sigma(-1)^i(2i+1)\,\sigma_5\{n-\tfrac{1}{2}i(i+1)\}$$
$$+\tfrac{10}{3}\Sigma(-1)^i i(i+1)(2i+1)\,\sigma_3\{n-\tfrac{1}{2}i(i+1)\}$$
$$+\tfrac{1}{3}\Sigma(-1)^i i(i+1)(2i+1)(3i^2+3i-1)\,\sigma\{n-\tfrac{1}{2}i(i+1)\} = 0.$$

For example, putting $n = 6$, and adopting the last of the three conventions, the theorem gives

$$\sigma_5(6)-3\sigma_5(5)+5\sigma_5(3)-7\sigma_5(0) = 20\left\{\sigma_3(5)-5\sigma_3(3)+14\sigma_3(0)\right\}$$
$$+10\left\{\sigma(5)-17\sigma(3)+98\sigma(0)\right\},$$

where $\sigma(0) = \frac{6}{7},\quad \sigma_3(0) = -\frac{6}{35},\quad \sigma_5(0) = \frac{3}{7}.$

The same Theorems expressed in terms of c and t. §§ 18, 19.

18. The coefficients in the series may conveniently be expressed by means of c, the coefficient, and t, the triangular number, which occur in the general term of the simplest of the series, viz.,

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+9\sigma(n-10)-\&c.$$

Let, therefore, t denote the i^{th} triangular number $\frac{1}{2}i(i+1)$, and let c denote the coefficient $2i+1$ of the term $\phi(n-t)$, so that the quantities c and t are connected by the relation

$$t = \tfrac{1}{8}(c^2-1).$$

Using these letters c and t, and extending the sign of summation to every triangular number which does not exceed n, zero being included, and also n itself, if n is not a triangular number, we may write the above series in the form

$$\Sigma(-1)^{\frac{1}{2}(c-1)}c\sigma(n-t);$$

or, if we employ Legendre's symbol $\left(\frac{-1}{c}\right)$ in place of its value $(-1)^{\frac{1}{2}(c-1)}$, in the form

$$\Sigma\left(\frac{-1}{c}\right)c\sigma(n-1).$$

Now it is known that

$$1^{2r}+2^{2r}+3^{2r}+\ldots+i^{2r} = ctF(t),$$

where $F(t)$ is a rational and integral function of t of the order $r-1$, so that the series

$$\phi(n-1)-(1^{2r}+2^{2r})\phi(n-3)+(1^{2r}+2^{2r}+3^{2r})\phi(n-6)-\&c.$$

may always be expressed in the form

$$-\Sigma\left(\frac{-1}{c}\right)ct\,F(t)\,\phi(n-t),$$

the limits of summation being the same as above.

19. Expressed in this notation, the formulæ of §§ 15–17 assume the forms

$$\Sigma\left(\frac{-1}{c}\right)c\sigma\,(n-t)=0,$$

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_3\,(n-t)+2\Sigma\left(\frac{-1}{c}\right)ct\,\sigma\,(n-t)=0,$$

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_5\,(n-t)+\tfrac{20}{3}\Sigma\left(\frac{-1}{c}\right)ct\,\sigma_3\,(n-t)+4\Sigma\left(\frac{-1}{c}\right)ct^3\,\sigma\,(n-t)$$
$$-\tfrac{2}{3}\Sigma\left(\frac{-1}{c}\right)ct\,\sigma\,(n-t)=0,$$

where, as in §§ 15–17, in the first formula, $\sigma\,(0)=\frac{1}{3}n$; in the second, $\sigma\,(0)=\frac{1}{5}n$, $\sigma_3\,(n)=-\frac{1}{15}n$; in the third, $\sigma\,(0)=\frac{1}{7}n$, $\sigma_3\,(0)=-\frac{1}{35}n$, $\sigma_5\,(0)=\frac{1}{21}n$.

It is evident that the general theorem in § 11 may be expressed as a linear relation connecting together series of the form

$$\Sigma\left(\frac{-1}{c}\right)ct^r\,\sigma_s\,(n-t),$$

where r is any number, and s any uneven number.

Method of representing the Additional Term in the general Theorem. §§ 20–23.

20. The most elegant of the different methods which have been made use of in the preceding sections for representing the additional term is the one in which the values of $\sigma\,(0)$, $\sigma_3\,(0)$, ... are simply proportional to n. It is easy to see that this method of representing the additional term is general. For, writing the general theorem (§ 11), when n is the triangular number $\frac{1}{2}g\,(g+1)$, in the form

$$\sigma_m\,(n)-3\sigma_m\,(n-1)+5\sigma_m\,(n-3)-7\sigma_m\,(n-6)+9\sigma_m\,(n-10)-\&c.$$
$$=\quad 2\frac{m^{(2)}}{2\,!}\left\{\sigma_{m-2}\,(n-1)-(1^2+2^2)\,\sigma_{m-2}\,(n-3)\right.$$
$$\left.+(1^2+2^2+3^2)\,\sigma_{m-2}\,(n-6)-\&c.\right\}$$
$$+2\frac{m^{(4)}}{4\,!}\left\{\sigma_{m-4}\,(n-1)-(1^4+2^4)\,\sigma_{m-4}\,(n-3)\right.$$
$$\left.+(1^4+2^4+3^4)\,\sigma_{m-4}\,(n-6)-\&c.\right\}$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$+2m\left\{\sigma\,(n-1)-(1^{m-1}+2^{m-1})\,\sigma\,(n-3)\right.$$
$$\left.+(1^{m-1}+2^{m-1}+3^{m-1})\,\sigma\,(n-6)-\&c.\right\}$$

[illegible] $^{-1}(1^{m+1}+2$ [illegible] $+\dots+g^{m+1}),$

where m is any uneven number, and $m^{(r)}$ denotes the factorial

$$m(m-1)\dots(m-r+1),$$

it is evident that, if $a_1, a_3, \dots, a_m$ are numerical quantities depending upon m, which are determined by the equation

$$\begin{aligned}1^{m+1}+2^{m+1}+3^{m+1}+\dots+g^{m+1} \\ = g(g+1)\Big\{&m(1^{m-1}+2^{m-1}+3^{m-1}+\dots+g^{m-1})\,a_1 \\ &+\frac{m^{(3)}}{3\,!}(1^{m-3}+2^{m-3}+3^{m-3}+\dots+g^{m-3})\,a_3 \\ &\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ &+\frac{m^{(2)}}{2\,!}(1^2+2^2+3^2+\dots+g^2)\,a_{m-2} \\ &+\tfrac{1}{2}(2g+1)\,a_m,\end{aligned}$$

then the additional term disappears if we assign to $\sigma(0)$, $\sigma_3(0)$, ... the system of values

$$\sigma(0)=a_1 n,\quad \sigma_3(0)=a_3 n,\quad \sigma_5(0)=a_5 n,\ \dots,\quad \sigma_m(0)=a_m n.$$

21. By replacing the series

$$1^{m+1}+2^{m+1}+3^{m+1}+\dots+g^{m+1},\quad 1^{m-1}+2^{m-1}+3^{m-1}+\dots+g^{m-1},\ \&c.$$

by their summations in terms of Bernoullian numbers, and equating coefficients, we ultimately arrive at the following simple system of equations, which determine the values of $a_1, a_3, \dots, a_m$*:—

$$ma_1 = 1-\frac{2}{m+2},$$

---

* The details of this determination are given in a paper "On a Theorem relating to Sums of Powers of the Natural Numbers" (*Messenger of Mathematics*, Vol. xx., pp. 120–128). The determination was made the subject of a separate paper, as the investigation is somewhat lengthy, and it is only with the results that the present paper is concerned. It is shown that, if the series $1^n+2^n+3^n+\dots+g^n$ be denoted by S_n, then

$$S_{2n} = S_1\left\{A_2 S_{2n-2}+A_4 S_{2n-4}+\dots+A_{2n-2} S_2 + A_{2n}(g+\tfrac{1}{2})\right\},$$

where $\quad \tfrac{1}{2}A_2 = ma_1,\quad \tfrac{1}{2}A_4 = \frac{m^{(3)}}{3\,!}a_3, \dots,\quad \tfrac{1}{2}A_{2n-2} = \frac{m^{(2)}}{2\,!}a_{m-2},\quad \tfrac{1}{2}A_{2n} = a_m,$

m being $=2n-1$, and $a_1, a_3, \dots, a_m$ having the values assigned to them in the text. In connexion with this investigation the theorems in § 11, which gave rise to it, are also given, but without proof, in a separate note (*Messenger*, Vol. xx., pp. 129–135).

$$\frac{m^{(3)}}{3\,!}a_3 - ma_1 \quad = -(m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5\,!}a_5 - \frac{m^{(3)}}{3\,!}a_3 = \quad \frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7\,!}a_7 - \frac{m^{(5)}}{5\,!}a_5 = -\frac{(m+1)^{(5)}}{5\,!}\frac{B_3}{3},$$

$$\&c., \qquad \&c.,$$

B_1, B_2, B_3, ... being the Bernoullian numbers.

We thus find

$$a_1 = \frac{1}{m+2},$$

$$\frac{m^{(3)}}{3\,!}a_3 = \frac{m}{m+2} - (m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5\,!}a_5 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7\,!}a_7 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2} - \frac{(m+1)^{(5)}}{5\,!}\frac{B_3}{3},$$

$$\&c., \qquad \&c.,$$

whence

$$\sigma(0) = \frac{1}{m+2}n,$$

$$\frac{m^{(3)}}{3\,!}\sigma_3(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1}\right\}n,$$

$$\frac{m^{(5)}}{5\,!}\sigma_5(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2}\right\}n,$$

$$\frac{m^{(7)}}{7\,!}\sigma_7(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2} - \frac{(m+1)^{(5)}}{5\,!}\frac{B_3}{3}\right\}n,$$

$$\&c., \qquad \&c.$$

Thus finally we obtain the formulæ:

$$\sigma(0) = \frac{1}{m+2}n,$$

$\frac{\sigma_3(0)}{m+1} = \Big\{\frac{3}{(m}$ [illegible] $\frac{B_1}{1}\Big\}$ [illegible]

$$\frac{\sigma_5(0)}{m+1} = \left\{ \frac{5!\,m}{(m+2)^{(7)}} - \frac{5!}{m^{(7)}}\frac{B_1}{1} + \frac{5^{(2)}}{(m-2)^{(3)}}\frac{B_2}{2} \right\} n,$$

$$\frac{\sigma_7(0)}{m+1} = \left\{ \frac{7!\,m}{(m+2)^{(9)}} - \frac{7!}{m^{(7)}}\frac{B_1}{1} + \frac{7^{(4)}}{(m-2)^{(5)}}\frac{B_2}{2} - \frac{7^{(2)}}{(m-4)^{(3)}}\frac{B_3}{3} \right\} n,$$

and, in general, r being any uneven number,

$$\frac{\sigma_r(0)}{m+1} = \left\{ \frac{r!\,m}{(m+2)^{(r+2)}} - \frac{r!}{m^{(r)}}\frac{B_1}{1} + \frac{r^{(r-3)}}{(m-2)^{(r-2)}}\frac{B_2}{2} - \frac{r^{(r-5)}}{(m-4)^{(r-4)}}\frac{B_3}{3} + \dots \right.$$

$$\left. \dots \pm \frac{r^{(2)}}{(m-r+3)^{(3)}}\frac{B_{\frac{1}{2}(r-1)}}{\frac{1}{2}(r-1)} \right\} n.$$

22. These equations give

$$\sigma\,(0) = \frac{1}{m+2}\,n,$$

$$\sigma_3(0) = -\frac{1}{(m+2)\,m}\,n,$$

$$\sigma_5(0) = \frac{3m+10}{3\,(m+2)\,m\,(m-2)}\,n,$$

$$\sigma_7(0) = -\frac{m^2+12m+56}{3\,(m+2)\,m\,(m-2)(m-4)}\,n,$$

&c., &c.

The values obtained by putting $m = 3$ and 5 in these expressions agree with the third system of results in §§ 16 and 17.

Putting $m = 7$, we find

$$\sigma(0) = \tfrac{1}{9}n, \quad \sigma_3(0) = -\tfrac{1}{63}n, \quad \sigma_5(0) = \tfrac{17}{945}n, \quad \sigma_7(0) = -\tfrac{1}{15}n.$$

If, therefore, we attribute these values to $\sigma(0)$, $\sigma_3(0), \dots, \sigma_7(0)$, we have, for all values of n,

$$\sigma_7(n) - 3\sigma_7(n-1) + 5\sigma_7(n-3) - 7\sigma_7(n-6) + 9\sigma_7(n-10) - \&c.$$

$$= 42\,\{\sigma_5(n-1) - (1^2+2^2)\,\sigma_5(n-3) + (1^2+2^2+3^2)\,\sigma_5(n-6) - \&c.\}$$

$$+ 70\,\{\sigma_3(n-1) - (1^4+2^4)\,\sigma_3(n-3) + (1^4+2^4+3^4)\,\sigma_3(n-6) - \&c.\}$$

$$+ 14\,\{\sigma\,(n-1) - (1^6+2^6)\,\sigma\,(n-3) + (1^6+2^6+3^6)\,\sigma\,(n-6) - \&c.\}.$$

23. With respect to the two methods of presenting the general theorem (§§ 11 and 20), *i.e.*, with or without the additional term, it is evidently an advantage to avoid the irregularity produced in the formula by the occasional presence of an extra term. Since the

quantities $\sigma(0)$, $\sigma_3(0)$, ..., $\sigma_m(0)$ only enter into the formula in the exceptional case when the additional term makes its appearance, it seems natural, following Euler,* to get rid of this term by assigning suitable values to these quantities. It is interesting to find that this can always be effected by means of a system of values, all of which are simply proportional to n. The form of the expressions for these values in terms of Bernoullian numbers (§ 21) is also not without interest. On the other hand, the simplicity and directness of the theorem, as stated in § 11, is impaired by these conventional meanings assigned to $\sigma(0)$, $\sigma_3(0)$, ..., $\sigma_m(0)$. Also, the additional term itself has a special interest of its own, being similar in form (*i.e.*, a sum of even powers of the natural numbers) to the coefficients in the series.

Form of the General Theorem if the Divisor unity be omitted. § 24.

24. It may be remarked that, if we omit from the divisors of n, $n-1$, $n-3$, ... the divisor unity which is common to them all, the only change introduced into the formulæ is the addition of a single term consisting of an uneven power of l, where $\frac{1}{2}l(l+1)$ is the triangular number nearest to, and not inferior to, n.

Thus, if $\sigma'_r(n)$ denotes the sum of the r^{th} powers of the divisors of n, unity being included, we have, corresponding to the general theorem in § 11,

$$\sigma'_m(n)-3\sigma'_m(n-1)+5\sigma'_m(n-3)-7\sigma'_m(n-6)+9\sigma'_m(n-10)-\&c.$$

$$\begin{aligned}
= \quad & 2\frac{m^{(2)}}{2!}\{\sigma'_{m-2}(n-1)-(1^2+2^2)\,\sigma'_{m-2}(n-3) \\
& \qquad\qquad +(1^2+2^2+3^2)\,\sigma'_{m-2}(n-6)-\&c.\} \\
& +2\frac{m^{(4)}}{4!}\{\sigma'_{m-4}(n-1)-(1^4+2^4)\,\sigma'_{m-4}(n-3) \\
& \qquad\qquad +(1^4+2^4+3^4)\,\sigma'_{m-2}(n-6)-\&c.\} \\
& \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
& +2m\{\sigma'(n-1)-(1^{m-1}+2^{m-1})\,\sigma'(n-3) \\
& \qquad\qquad +(1^{m-1}+2^{m-1}+3^{m-1})\,\sigma'(n-6)-\&c.\} \\
& +(-1)^l l^m+\left[(-1)^{g-1}(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1})\right],
\end{aligned}$$

---

* In Euler's recurring formula

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\&c.=0,$$

he dispensed with an additional term which occurs when n is a pentagonal number by assigning to $\sigma(0)$ the value n (*Opera Minora Collecta*, Vol. I., p. 149).

where l has the meaning just assigned to it, so that, when n is a triangular number $\frac{1}{2}g\,(g+1)$, in which case the additional term appears, l is equal to g; otherwise $\frac{1}{2}l\,(l+1)$ is the triangular number next greater than n. It will be observed that l is always equal to the number of terms in the series on the left-hand side of the equation, not counting the term $\sigma_m\,(0)$, if it occurs. As examples, let $m = 3$, and put $n = 5$ and 6. In these two cases the formula becomes

$$\sigma_3'\,(5)-3\sigma_3'\,(4)+5\sigma_3'\,(2)=6\left\{\sigma'\,(4)-5\sigma'\,(2)\right\}-3^3,$$

and $$\sigma_3'\,(6)-3\sigma_3'\,(5)+5\sigma_3'\,(3)=6\left\{\sigma'\,(5)-5\sigma'\,(3)\right\}-3^3+1^4+2^4+3^4;$$

that is, $$125-3\times\ 72+5\times\ 8=6\left\{6-10\right\}-27,$$

and $$251-3\times125+5\times27=6\left\{5-15\right\}-27+98.$$

Proof of the Theorem relating to the actual Divisors. §§ 25–27.

25. Let $$\text{zs}\,x=\frac{\Theta_1'\,(x)}{\Theta_1\,(x)}=\text{zn}\,x+\frac{\text{cn}\,x\,\text{dn}\,x}{\text{sn}\,x},$$

where $\Theta_1\,(x)$ and $\text{zn}\,x$ are the same as Jacobi's $\text{H}\,(x)$ and $\text{Z}\,(x)$ respectively. Then, denoting $\dfrac{2K}{\pi}$ by ρ, we have

$$\rho\,\text{zs}\,\rho x=\frac{2q^{\frac{1}{4}}\cos x-6q^{\frac{9}{4}}\cos 3x+10q^{\frac{25}{4}}\cos 5x-\&\text{c.}}{2q^{\frac{1}{4}}\sin x-2q^{\frac{9}{4}}\sin 3x+2q^{\frac{25}{4}}\sin 5x-\&\text{c.}}.$$

Now, it can be shown that

$$\rho\,\text{zs}\,\rho x=\frac{\cos x}{\sin x}+\frac{4q^2}{1-q^2}\sin 2x+\frac{4q^4}{1-q^4}\sin 4x+\frac{4q^6}{1-q^6}\sin 6x+\&\text{c.}$$

Let $\sigma\phi\,(n)$ denote the sum

$$\phi\,(d_1)+\phi\,(d_2)+\phi\,(d_3)+\ldots+\phi\,(d_f),$$

where $d_1, d_2, d_3, \ldots, d_f$ are all the divisors of n. Using this notation, the coefficient of q^{2n} in the above q-series is easily seen to be equal to $4\sigma\,(\sin 2nx)$, and the equation may be written

$$\rho\,\text{zs}\,\rho x=\frac{\cos x}{\sin x}+4\Sigma_1^\infty\,\sigma\,(\sin 2nx)\,q^{2n}.\text{[*]}$$

---

[*] This equation was given in this form in the *Messenger of Mathematics*, Vol. XVIII., p. 6, as one of a system of sixteen similar formulæ representing the twelve elliptic and four zeta functions. It is to be observed that the symbol σ is supposed always to refer to the divisors of n (not $2n$). The sign of summation also refers to n.

Equating the two expressions for $\rho\,\mathrm{zs}\,\rho x$, we obtain the identical relation

$$\frac{\cos x}{\sin x}+4\Sigma_1^\infty\sigma(\sin 2nx)\,q^{2n}=\frac{\cos x-3q^2\cos 3x+5q^6\cos 5x-\&c.}{\sin x-q^2\sin 3x+q^6\sin 5x-\&c.}.$$

26. Replacing q^2 by q, this equation becomes

$$\frac{\cos x}{\sin x}+4\Sigma_1^\infty\sigma(\sin 2nx)\,q^{n}=\frac{\cos x-3q\cos 3x+5q^3\cos 5x-\&c.}{\sin x-q\sin 3x+q^3\sin 5x-\&c.};$$

whence, by multiplying up, we find

$$\begin{aligned}4(\sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+\&c.)\,\Sigma_1^\infty\sigma(\sin 2nx)\,q^n\\ =\quad &(\sin 3x\cot x-3\cos 3x)\,q\\ &-(\sin 5x\cot x-5\cos 5x)\,q^3\\ &+(\sin 7x\cot x-7\cos 7x)\,q^6\\ &-\dots\dots\dots\dots\dots\end{aligned}$$

$$=-\sin x\frac{d}{dx}\frac{\sin 3x}{\sin x}q+\sin x\frac{d}{dx}\frac{\sin 5x}{\sin x}q^3-\sin x\frac{d}{dx}\frac{\sin 7x}{\sin x}q^6-\&c.$$

The equation may therefore be expressed in the form

$$4\left(1-\frac{\sin 3x}{\sin x}q+\frac{\sin 5x}{\sin x}q^3-\frac{\sin 7x}{\sin x}q^6+\&c.\right)\Sigma_1^\infty\sigma(\sin 2nx)\,q^n$$

$$=-\frac{d}{dx}\frac{\sin 3x}{\sin x}q+\frac{d}{dx}\frac{\sin 5x}{\sin x}q^3-\frac{d}{dx}\frac{\sin 7x}{\sin x}q^6+\&c.^*$$

Now
$$\frac{\sin 3x}{\sin x}=1+2\cos 2x,$$

$$\frac{\sin 5x}{\sin x}=1+2\cos 2x+2\cos 4x,$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots;$$

and, in general, m being any uneven number,

$$\frac{\sin mx}{\sin x}=1+2\cos 2x+2\cos 4x+\dots+2\cos(m-1)\,x.$$

---

[*] This equation may also [illegible] very simply by writing the first equation in § 26 in the form

$$4\Sigma_1^\infty\sigma(\sin 2nx)\,q^n+\text{[illegible]}(\sin\text{[illegible]}\sin 5x-\&c.).$$

The equation therefore becomes

$$\{1-(1+2\cos 2x)\,q+(1+2\cos 2x+2\cos 4x)\,q^3-\&c.\}\,\Sigma_1^\infty\,\sigma(\sin 2nx)\,q^n$$

$$=\sin 2x\,.\,q-(\sin 2x+2\sin 4x)\,q^3+(\sin 2x+2\sin 4x+3\sin 6x)\,q^6-\&c.$$

27. Replacing $2x$ by x, we have finally

$$\{1-(1+2\cos x)\,q+(1+2\cos x+2\cos 2x)\,q^3$$

$$-(1+2\cos x+2\cos 2x+2\cos 3x)\,q^6+\&c.\}$$

$$\times\,\{\Sigma_1\sin dx\,.\,q+\Sigma_2\sin dx\,.\,q^2+\Sigma_3\sin dx\,.\,q^3+\&c.\}$$

$$=\sin x\,.\,q-(\sin x+2\sin 2x)\,q^3+(\sin x+2\sin 2x+3\sin 3x)\,q^6-\&c.,$$

where, in accordance with the notation employed in § 9, $\Sigma_n\phi(d)$ denotes $\phi(d_1)+\phi(d_2)+\ldots+\phi(d_r)$; $d_1, d_2, \ldots, d_r$ being all the divisors of n.

The coefficient of q^n on the left-hand side of this equation is

$$\Sigma_n\sin dx-\Sigma_{n-1}\{\sin dx+\sin(d-1)\,x+\sin(d+1)\,x\}$$

$$+\Sigma_{n-3}\{\sin dx+\sin(d-1)\,x+\sin(d+1)\,x+\sin(d-2)\,x+\sin(d+2)\,x\}$$

$$-\&c.$$

We see, therefore, by equating coefficients, that this expression must be equal to zero if n is not a triangular number, and that it is equal to

$$(-1)^{g-1}(\sin x+2\sin 2x+3\sin 3x+\ldots+g\sin gx)$$

when n is the triangular number $\tfrac{1}{2}g\,(g+1)$.

It is evident that this relation can only exist by virtue of the actual numbers d, $d-1$, $d+1$, &c., which occur in the arguments on the left-hand side, cancelling each other (with the exception of the one 1, two 2's, three 3's, ..., which are to remain when n is a triangular number); for no property of the sine, as distinguished from any other uneven function, can have any influence upon the mutual destruction of the terms. We may, therefore, either replace the sines by ϕ's, ϕ being any uneven function, thus obtaining the theorem in § 9, or we may pass directly from the sine-theorem to the general theorem relating to the actual divisors (§ 3).

Second Theorem relating to the Actual Divisors of the Numbers n, $n-1$, $n-3$, *&c.* §§ 28–30.

28. From the last formula in § 26, by multiplication by $\sin x$, we find

$$2\,(\sin x - q\sin 3x + q^3 \sin 5x - q^6 \sin 7x + \&c.)\ \Sigma_1^\infty \sigma (\sin 2nx)\, q^n$$

$$= (\cos x - \cos 3x)\, q - (\cos x + \cos 3x - 2\cos 5x)\, q^3$$
$$+ (\cos x + \cos 3x + \cos 5x - 3\cos 7x)\, q^6 - \&c.,$$

the general term on the right-hand side being

$$\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2g-1)\,x - g\cos(2g+1)\,x\}\ q^{\frac{1}{2}g(g+1)}.$$

The coefficient of q^n on the left-hand side is

$$\begin{aligned}
&\Sigma_n \{\cos(2d-1)\,x - \cos(2d+1)\,x\} \\
&-\Sigma_{n-1}\{\cos(2d-3)\,x - \cos(2d+3)\,x\} \\
&+\Sigma_{n-3}\{\cos(2d-5)\,x - \cos(2d+5)\,x\} \\
&-\Sigma_{n-6}\{\cos(2d-7)\,x - \cos(2d+7)\,x\} \\
&+\&c.;
\end{aligned}$$

this expression therefore is equal to zero, if n is not a triangular number, and is equal to

$$(-1)^{g-1}\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2g-1)\,x - g\cos(2g+1)\,x\},$$

when n is a triangular number $\frac{1}{2}g\,(g+1)$.

29. Using the notation of § 5 so that $\alpha, \beta, \ldots$, are the divisors of n, $\alpha_1, \beta_1, \ldots$, those of $n-1$, $\alpha_3, \beta_3, \ldots$, of $n-3$, &c., it follows that the numbers given by the formula,

$$\left\{\begin{matrix} 2\alpha+1, & 2\beta+1, & \ldots \\ -(2\alpha-1), & -(2\beta-1), & \ldots \end{matrix}\right\} - \left\{\begin{matrix} 2\alpha_1+3, & 2\beta_1+3, & \ldots \\ -[2\alpha_1-3], & -[2\beta_1-3], & \ldots \end{matrix}\right\}$$
$$+\left\{\begin{matrix} 2\alpha_3+5, & 2\beta_3+5, & \ldots \\ -[2\alpha_3-5], & -[2\beta_3-5], & \ldots \end{matrix}\right\} - \&c.,$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g\,(g+1)$, in which case there are left remaining one 1, one 3, ..., one $2g-1$, all having the same sign, and also the number $2g+1$ occurring g times, and having the opposite sign. If g be even, the g different numbers have the positive sign; if uneven, the negative sign.

The numbers $2\alpha_1-3$, $2\beta_1-3$, ... are enclosed in square brackets to indicate that their absolute values are to be taken (irrespective of sign), *i.e.* $[a]$ denotes a, if a is positive, and $-a$ if a is negative, so that $[a]$ is always a positive quantity. It is unnecessary to enclose $2\alpha-1$, $2\beta-1$, ... in square brackets, as they are necessarily positive.*

As an example, let $n=6$. The doubles of the divisors of 6, 5, 3 are 2, 4, 6, 12; 2, 10; 2, 6 respectively, and, since $g=3$, the theorem asserts that the numbers

$$\left\{\begin{matrix} 3, & 5, & 7, & 13 \\ -1, & -3, & -5, & -11 \end{matrix}\right\} - \left\{\begin{matrix} 5, & 13 \\ -1, & -7 \end{matrix}\right\} + \left\{\begin{matrix} 7, & 11 \\ -3, & -1 \end{matrix}\right\}$$

cancel each other with the exception of -1, -3, -5, 7, 7, 7.

30. The theorem may be exhibited conveniently in the following manner.

Taking the above example, we first write down in a central line the doubles of the divisors 6, 5, 3. We add 1 to each of the divisors in the first set, writing the numbers so obtained above them; we then add 3 to the second set, writing the numbers below; then 5 to the next set, writing the numbers above. We have thus formed the scheme:

$$\begin{array}{ccc} 3,\ 5,\ 7,\ 13, & & 7,\ 11, \\ (2,\ 4,\ 6,\ 12), & (2,\ 10), & (2,\ \ 6), \\ & 5,\ 13. & \end{array}$$

To complete the scheme, we subtract 1 from each of the divisors in the first set, 3 from the second set, 5 from the third set, writing the numbers below and above alternately, and attending only to the absolute values of the numbers (for example, in the second set, subtracting 3 from 2 and 10 we obtain -1 and 7, and we enter 1

---

* In the notation of § 3, the theorem asserts that the numbers given by the formula,

$$G_n\begin{pmatrix} 2d+1 \\ -[2d-1] \end{pmatrix} - G_{n-1}\begin{pmatrix} 2d+3 \\ -[2d-3] \end{pmatrix} + G_{n-3}\begin{pmatrix} 2d+5 \\ -[2d-5] \end{pmatrix} - G_{n-6}\begin{pmatrix} 2d+7 \\ -[2d-7] \end{pmatrix} + \&c.,$$

all cancel each other if n is not a triangular number, and reduce to

$$(-1)^g \{1,\ 3,\ 5,\ \ldots,\ 2g-1, \text{ and } -(2g+1) \text{ occurring } g \text{ times}\},$$

if n is the g^{th} triangular number $\frac{1}{2}g\,(g+1)$.

and 7). We thus obtain the system of numbers

$$\begin{array}{ccc} 3,\ 5,\ 7,\ 13, & 1,\ 7, & 7,\ 11, \\ (2,\ 4,\ 6,\ 12), & (2,\ 10), & (2,\ 6), \\ 1,\ 3,\ 5,\ 11, & 5,\ 13, & 3,\ 1. \end{array}$$

The theorem then asserts that, if we cancel the numbers which occur both in the upper and lower lines (ignoring the middle line), we shall have left three 7's in the upper line, and the three uneven numbers inferior to 7 (i.e. 1, 3, 5) in the lower line.

In general, if the numbers derived from the divisors be written down according to this method, the theorem asserts that, if we regard the numbers occurring both in the upper and lower lines as cancelling each other, then this cancellation is complete if n is not a triangular number, but that when n is equal to a triangular number $\frac{1}{2}g(g+1)$, then there are left, after the cancelling, g $(2g+1)$'s in one line, and the g uneven numbers inferior to $2g+1$ in the other. If g is even, the $(2g+1)$'s are left in the lower line; if uneven, in the upper line.

Taking as examples the numbers 9 and 10 (as in § 4), and forming the systems of numbers, we obtain, for $n = 9$,

$$\begin{array}{cccc} 3,\ 7,\ 19, & 1,\ 1,\ 5,\ 13, & 7,\ 9,\ 11,\ 17, & 5,\ 1, \\ (2,\ 6,\ 18), & (2,\ 4,\ 8,\ 16), & (2,\ 4,\ 6,\ 12), & (2,\ 6), \\ 1,\ 5,\ 17, & 5,\ 7,\ 11,\ 19, & 3,\ 1,\ 1,\ 7, & 9,\ 13, \end{array}$$

in which all the numbers in the upper and lower lines cancel each other, and, for $n = 10$,

$$\begin{array}{cccc} 3,\ 5,\ 11,\ 21, & 1,\ 3,\ 15, & 7,\ 19, & 5,\ 3,\ 1, \\ (2,\ 4,\ 10,\ 20), & (2,\ 6,\ 18), & (2,\ 14), & (2,\ 4,\ 8), \\ 1,\ 3,\ 9,\ 19, & 5,\ 9,\ 21, & 3,\ 9, & 9,\ 11,\ 15, \end{array}$$

in which, after cancelling all the numbers common to the upper and lower lines, we have left four 9's in the lower line, and the four uneven numbers inferior to 9, viz., 1, 3, 5, 7, in the upper line.

Numerical Theorems relating to the Sums of Powers of the Divisors of n, $n-1$, $n-3$, &c. §§ 31–33.

31. By replacing each of the numbers in the theorem of § 29 by the same arbitrary function of itself and adding together the members of

each group, as in § 9, we find that, if ψ be any even function, then

$$\Sigma_n \{\psi(2d+1) - \psi(2d-1)\} - \Sigma_{n-1} \{\psi(2d+3) - \psi(2d-3)\}$$
$$+ \Sigma_{n-3} \{\psi(2d+5) - \psi(2d-5)\} - \Sigma_{n-6} \{\psi(2d+7) - \psi(2d-7)\}$$
$$+ \&c.$$

is equal to zero, if n is not a triangular number, and equal to

$$(-1)^{g-1} \{g\psi(2g+1) - \psi(1) - \psi(3) - \psi(5) - \ldots - \psi(2g-1)\}$$

if n is a triangular number $\frac{1}{2}g(g+1)$.

This result may of course be deduced directly from the theorem in § 28, by replacing the cosines by arbitrary functions.

32. As a particular case, let

$$\psi(2d+1) = (2d+1)^2.$$

We thus find

$$\Sigma_n 8d - \Sigma_{n-1} 24d + \Sigma_{n-3} 40d - \Sigma_{n-6} 56d + \&c.$$
$$= 0, \quad \text{or} \quad (-1)^{g-1} \{g(2g+1)^2 - 1^2 - 3^2 - 5^2 - \ldots - (2g-1)^2\},$$

according as n is not a triangular number, or is equal to $\frac{1}{2}g(g+1)$.

Now $$1^2 + 3^2 + 5^2 + \ldots + (2g-1)^2 = \tfrac{1}{3}g(2g-1)(2g+1),$$

so that $g(2g+1)^2 - 1^2 - 3^2 - 5^2 - \ldots - (2g-1)^2 = \frac{4}{3}g(g+1)(2g+1)$,

and the formula becomes

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + \&c.$$
$$= 0, \text{ or } (-1)^{g-1} \tfrac{1}{6}g(g+1)(2g+1),$$

according as n is not a triangular number, or is equal to $\frac{1}{2}g(g+1)$. This result is equivalent to the recurring formula quoted in the first section of this paper.

33. Putting $$\psi(2d+1) = (2d+1)^{m+1},$$

m being any uneven number, we obtain the general theorem

$$\sigma_m(n) - 3\sigma_m(n-1) + 5\sigma_m(n-3) - 7\sigma_m(n-6) + \&c.$$
$$+ \frac{m^{(2)}}{2^2.3!} \{\sigma_{m-2}(n) - 3^3\sigma_{m-2}(n-1) + 5^3\sigma_{m-2}(n-3) - 7^3\sigma_{m-2}(n-6) + \&c.\}$$
$$+ \frac{m^{(4)}}{2^4.5!} \{\sigma_{m-4}(n) - 3^5\sigma_{m-4}(n-1) + 5^5\sigma_{m-4}(n-3) - 7^5\sigma_{m-4}(n-6) + \&c.\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+\frac{m^{(2)}}{2^{m-3}.3!}\{\sigma_3(n)-3^{m-2}\sigma_3(n-1)+5^{m-2}\sigma_3(n-3)-7^{m-2}\sigma_3(n-6)+\&c.\}$$

$$+\frac{1}{2^{m-1}}\{\sigma(n)-3^{m}\sigma(n-1)+5^{m}\sigma(n-3)-7^{m}\sigma(n-6)+\&c.\}$$

$$=\Big[(-1)^{g-1}\frac{1}{2^{m+1}(m+1)}\{g(2g+1)^{m+1}-1^{m+1}-3^{m+1}-5^{m+1}-\dots$$
$$\dots-(2g-1)^{m+1}\}\Big],$$

where (as in § 11) the square brackets denote that the term enclosed by them only appears when n is equal to $\frac{1}{2}g(g+1)$. Otherwise the right-hand side of the equation is zero.

This result corresponds to the general theorem in § 11. Comparing the two formulæ, we see that in § 11 all the series except the first have one term less, and that the additional term is somewhat simpler in form. On the other hand, the coefficients in the above series are much simpler than in § 11, consisting each of a single power, and all the series are of exactly the same form, whereas in § 11 the first series, on the left-hand side of the equation, is not included in the general form of those on the right-hand side.

The particular cases of $m=3$ and 5. §§ 34–37.

34. By putting $m = 3$ and 5, we find

$$\sigma_3(n)-3\sigma_3(n-1)+5\sigma_3(n-3)-7\sigma_3(n-6)+\&c.$$
$$+\tfrac{1}{4}\{\sigma(n)-3^3\sigma(n-1)+5^3\sigma(n-3)-7^3\sigma(n-6)+\&c.\}$$
$$=\Big[(-1)^{g-1}\frac{1}{2^4.4}\{g(2g+1)^4-1^4-3^4-5^4-\dots-(2g-1)^4\}\Big],$$

and

$$\sigma_5(n)-3\sigma_5(n-1)+5\sigma_5(n-3)-7\sigma_5(n-6)+\&c.$$
$$+\tfrac{5}{6}\{\sigma_3(n)-3^3\sigma_3(n-1)+5^3\sigma_3(n-3)-7^3\sigma_3(n-6)+\&c.\}$$
$$+\tfrac{1}{16}\{\sigma(n)-3^5\sigma(n-1)+5^5\sigma(n-3)-7^5\sigma(n-6)+\&c.\}$$
$$=\Big[(-1)^{g-1}\frac{1}{2^6.6}\{g(2g+1)^6-1^6-3^6-5^6-\dots-(2g-1)\}^6\Big].$$

35. Since

$$1^4+3^4+5^4+\dots+(2g-1)^4=\tfrac{16}{5}g^5-\tfrac{8}{3}g^3+\tfrac{7}{15}g.$$

we find that the additional term in the first of these two equations is equal to

$$(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{5}{12}g^3+\tfrac{1}{8}g^2+\tfrac{1}{120}g\right\},$$

which $$=(-1)^{g-1}\,\frac{(2g+1)(g^2+g)(12g^2+12g+1)}{120}.$$

In a similar manner, since

$$1^6+3^6+5^6+\ldots+(2g-1)^6=\tfrac{64}{7}g^7-16g^5+\tfrac{28}{3}g^3-\tfrac{31}{21}g,$$

we find that the additional term in the second equation

$$=(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{2}{3}g^5+\tfrac{5}{12}g^4+\tfrac{19}{144}g^3+\tfrac{1}{32}g^2+\tfrac{13}{1008}g\right\}$$

$$=(-1)^{g-1}\,\frac{(2g+1)(g^2+g)\left\{144\,(g^2+g)^2+24\,(g^2+g)+13\right\}}{2016}.$$

36. Expressed in the notation of §§ 16 and 17, the formulæ become

$$\Sigma\,(-1)^i\,(2i+1)\,\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}+\tfrac{1}{4}\Sigma\,(-1)^i\,(2i+1)^3\sigma\left\{n-\tfrac{1}{2}i(i+1)\right\}$$
$$=\left[(-1)^{g-1}\,\tfrac{1}{120}g\,(g+1)(2g+1)(12g^2+12g+1)\right],$$

and

$$\Sigma\,(-1)^i\,(2i+1)\,\sigma_5\left\{n-\tfrac{1}{2}i(i+1)\right\}+\tfrac{5}{6}\Sigma\,(-1)^i\,(2i+1)^3\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}$$
$$+\tfrac{1}{16}\Sigma\,(-1)^i\,(2i+1)^5\,\sigma\left\{n-\tfrac{1}{2}i\,(i+1)\right\}$$
$$=\left[(-1)^{g-1}\tfrac{1}{2016}\,g\,(g+1)(2g+1)\left\{144\,(g^2+g)^2+24\,(g^2+g)+13\right\},\right.$$

the summations extending from $i=0$ to $i=h$, where $\frac{1}{2}h\,(h+1)$ is the triangular number next inferior to n.

We may evidently dispense with the additional term (the right-hand member of the equations being then zero for all values of n) if we put, in the first formula,

$$\sigma\,(0)=0,\quad \sigma_3\,(0)=\tfrac{2}{5}n^2+\tfrac{1}{60}n\,;$$

and, in the second formula,

$$\sigma\,(0)=0,\quad \sigma_3\,(0)=0,\quad \sigma_5\,(0)=\tfrac{4}{7}n^3+\tfrac{1}{21}n^2+\tfrac{13}{1008}n.$$

If these terms of zero argument are included, the summations are to extend from $i=0$ to $i=\frac{1}{2}k\,(k+1)$, where $\frac{1}{2}k\,(k+1)$ is the triangular number, nearest to, and not exceeding, n.

37. We may, however, represent the additional term in a more elegant manner (as in §§ 16 and 17) by means of values of $\sigma\,(0)$, $\sigma_3\,(0)$, $\sigma_5\,(0)$, ..., all of which are simply proportional to n. These

values are found to be, for the first formula,

$$\sigma(0) = \tfrac{1}{5}n, \quad \sigma_3(0) = -\tfrac{1}{30}n,$$

and, for the second formula,

$$\sigma(0) = \tfrac{1}{7}n, \quad \sigma_3(0) = -\tfrac{1}{70}n, \quad \sigma_5(0) = \tfrac{1}{63}n.$$

The same Theorems expressed in terms of c and t. §§ 38–40.

38. In the notation of § 18, in which

$$c = 2i+1, \quad t = \tfrac{1}{2}i(i+1),$$

we may express the theorems, in the cases of $m = 1, 3, 5$, in the forms

$$\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + \tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c^3\sigma(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_5(n-t) + \tfrac{5}{6}\Sigma\left(\frac{-1}{c}\right) c^3\sigma_3(n-t) + \tfrac{1}{16}\Sigma\left(\frac{-1}{c}\right) c^5\sigma(n-t) = 0;$$

where, in the first formula, $\sigma(0) = \tfrac{1}{3}n$; in the second, $\sigma(0) = \tfrac{1}{5}n$, $\sigma_3(n) = -\tfrac{1}{30}n$; and in the third, $\sigma(0) = \tfrac{1}{7}n$, $\sigma_3(0) = -\tfrac{1}{70}n$, $\sigma_5(0) = \tfrac{1}{63}n$.

39. It may be observed that the formulæ contained in the general theorems of §§ 11 and 33 are not independent. Thus, writing the formulæ with the additional term, we have, from §§ 16 and 35,

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + 2\Sigma\left(\frac{-1}{c}\right) ct\sigma(n-t)$$
$$= \left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5 + \tfrac{1}{2}g^4 + \tfrac{1}{3}g^3 - \tfrac{1}{30}g\right\}\right],$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + \tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c^3\sigma(n-t)$$
$$= \left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5 + \tfrac{1}{2}g^4 + \tfrac{5}{12}g^3 + \tfrac{1}{8}g^2 + \tfrac{1}{120}g\right\}\right];$$

whence, by subtraction, since $c^2 = 8t+1$, we obtain

$$\tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = \left[(-1)^{g-1}\left\{\tfrac{1}{12}g^3 + \tfrac{1}{8}g^2 + \tfrac{1}{24}g\right\}\right];$$

that is, $\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = \left[(-1)^{g-1}\left\{\tfrac{1}{3}g^3 + \tfrac{1}{2}g^2 + \tfrac{1}{6}g\right\}\right]$,

which is the original recurring formula of § 1.

40. Similarly, from §§ 17 and 19, we have

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_5(n-t)+\tfrac{20}{3}\Sigma\left(\frac{-1}{c}\right)ct\sigma_3(n-t)+4\Sigma\left(\frac{-1}{c}\right)(ct^3-\tfrac{1}{6}ct)\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{1}{2}g^5-\tfrac{1}{6}g^3+\tfrac{1}{42}g\right\}\right],$$

and, from §§ 35 and 38,

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_5(n-t)+\tfrac{5}{6}\Sigma\left(\frac{-1}{c}\right)c(8t+1)\sigma_3(n-t)$$
$$+\tfrac{1}{16}\Sigma\left(\frac{-1}{c}\right)c(64t^2+16t+1)\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{2}{3}g^5+\tfrac{5}{12}g^4+\tfrac{19}{144}g^3+\tfrac{1}{32}g^2+\tfrac{13}{1008}g\right\}\right];$$

giving, by subtraction,

$$\tfrac{5}{6}\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+\tfrac{5}{3}\Sigma\left(\frac{-1}{c}\right)(ct+\tfrac{3}{80}c)\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{6}g^5+\tfrac{5}{12}g^4+\tfrac{43}{144}g^3+\tfrac{1}{32}g^2-\tfrac{5}{288}g\right\}\right];$$

that is,

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+2\Sigma\left(\frac{-1}{c}\right)(ct+\tfrac{3}{80}c)\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{43}{20}g^3+\tfrac{3}{80}g^2-\tfrac{1}{48}g\right\}\right].$$

Now $\tfrac{3}{40}\Sigma\left(\frac{-1}{c}\right)c\sigma(n-t)=\left[(-1)^{g-1}\left\{\tfrac{1}{40}g^3+\tfrac{3}{80}g^2+\tfrac{1}{80}g\right\}\right];$

whence, subtracting, we find

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+2\Sigma\left(\frac{-1}{c}\right)ct\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{1}{3}g^3-\tfrac{1}{30}g\right\}\right]$$
$$=\left[(-1)^{g-1}\left\{1^4+2^4+3^4+\ldots+g^4\right\}\right],$$

which is equivalent to the second formula of § 19.

Method of representing the Additional Term by constant values of $\sigma(0)$, $\sigma_3(0)$, &c. §§ 41–43.

41. The additional term in the general theorem of § 33 may be represented in the following manner:—

Let $f = 2g+1$, so that f is the coefficient of $\sigma_m(0)$, when it occurs. The additional term may then be written

$$\left[(-1)^{\frac{1}{2}(f+1)}\frac{1}{2^{m+1}(m+1)}\{\tfrac{1}{2}(f+1)f^{m+1}-1^{m+1}-3^{m+1}-5^{m+1}-\dots-f^{m+1}\}\right].$$

Now the sum of the series $1^{m+1}+3^{m+1}+5^{m+1}+\dots+f^{m+1}$ is (see § 53)

$$\tfrac{1}{2}\frac{f^{m+2}}{m+2}+\tfrac{1}{2}f^{m+1}+(m+1)^{1}B_1f^{m}-2^{2}\frac{(m+1)^{3}}{3!}\frac{B_2}{2}f^{m-2}$$

$$+2^{4}\frac{(m+1)^{5}}{5!}\frac{B_3}{3}f^{m-4}-\dots\pm2^{m-1}(m+1)^{m}\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}f.$$

The additional term is therefore equal to

$$\left[(-1)^{\frac{1}{2}(f+1)}\left\{\frac{f^{m+2}}{2^{m+2}(m+2)}-B_1\frac{f^{m}}{2^{m+1}}+\frac{m^{2}}{3!}\frac{B_2}{2}\frac{f^{m-2}}{2^{m-1}}\right.\right.$$

$$\left.\left.-\frac{m^{4}}{5!}\frac{B_3}{3}\frac{f^{m-4}}{2^{m-3}}-\dots\pm\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}\frac{f}{2^{2}}\right\}\right].$$

By comparing this expression with the left-hand member of the formula in § 33, it is evident that we may represent all the terms after the first by putting

$$\sigma(0) = -\tfrac{1}{2}B_1,\quad \sigma_3(0) = \tfrac{1}{2}\frac{B_2}{2},\quad \sigma_5(0) = -\tfrac{1}{2}\frac{B_3}{3},\ \dots,$$

$$\sigma_m(0) = (-1)^{\frac{1}{2}(m+1)}\tfrac{1}{2}\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)},$$

the general value being

$$\sigma_{2r-1}(0) = (-1)^{r}\tfrac{1}{2}\frac{B_r}{r}.$$

When these constant values are assigned to $\sigma(0)$, $\sigma_3(0)$, ... $\sigma_m(0)$, the additional term reduces to the monomial expression

$$\left[(-1)^{g+1}\frac{(2g+1)^{m+2}}{2^{m+2}(m+2)}\right].$$

It is to be noticed that the above values of $\sigma(0)$, $\sigma_3(0)$, ... $\sigma_m(0)$ are absolute numerical constants, being independent both of g and m.

[illegible]. We may cause the additional term to disappear entirely by assigning to one of the quantities $\sigma(0)$, $\sigma_3(0)$, ... a value depending upon g and m, all the others retaining their constant values.

For example, the additional term disappears if the right-hand member of the equation being then always zero) if we put

$$\sigma(0) = -\tfrac{1}{4}B_1 + \frac{f^2}{8(m+2)} = \tfrac{1}{8}\left(\frac{f^2}{m+2} - \tfrac{1}{3}\right).$$

Since $f^2 - 1 = 4(g^2+g) = 8n$, this value of $\sigma(0)$ may be written

$$\sigma(0) = \frac{n}{m+2} - \tfrac{1}{24}\,\frac{m-1}{m+2}.$$

Thus, for $m = 1$, $\sigma(0) = \tfrac{1}{3}n$;

„ $m = 3$, $\sigma(0) = \tfrac{1}{5}n - \tfrac{1}{60}$;

„ $m = 5$, $\sigma(0) = \tfrac{1}{7}n - \tfrac{1}{42}$;

„ $m = 7$, $\sigma(0) = \tfrac{1}{9}n - \tfrac{1}{36}$;

&c., &c.

43. Taking the particular cases of $m = 3$ and 5 (§§ 34 and 37), we have, therefore, for all values of n,

$$\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.$$
$$+\tfrac{1}{4}\{\sigma(n) - 3^3\sigma(n-1) + 5^3\sigma(n-3) - 7^3\sigma(n-6) + \&c.\} = 0,$$

if $$\sigma(0) = \tfrac{1}{5}n - \tfrac{1}{60},\quad \sigma_3(0) = \tfrac{1}{240},$$

and $$\sigma_5(n) - 3\sigma_5(n-1) + 5\sigma_5(n-3) - 7\sigma_5(n-6) + \&c.$$
$$+\tfrac{5}{6}\{\sigma_3(n) - 3^3\sigma_3(n-1) + 5^3\sigma_3(n-3) - 7^3\sigma_3(n-6) + \&c.\}$$
$$+\tfrac{1}{16}\{\sigma(n) - 3^5\sigma(n-1) + 5^5\sigma(n-3) - 7^5\sigma(n-6) + \&c.\} = 0,$$

if $$\sigma(0) = \tfrac{1}{7}n - \tfrac{1}{42},\quad \sigma_3(0) = \tfrac{1}{240},\quad \sigma_5(0) = -\tfrac{1}{504}.$$[*]

Formula connecting the Sums of the Divisors of the First n Numbers. § 44.

44. In a paper[†] in the fifth volume of the *Proceedings of the Cambridge Philosophical Society*, I gave the following formula which

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[*] The general theorem in § 33 and the method of representing the additional term in §§ 41 and 42 are given without proof in the *Messenger*, Vol. xx., pp. 176–191, in connexion with the investigation referred to in the note to § 53.

[†] "On the Sum of the Divisors of a Number" (1884, p. 108). The formula is given on p. 112. It also occurs in *Proc. Lond. Math. Soc.*, Vol. xv., p. 118 (1884).

connects together the sums of the divisors of all the numbers from unity to n:

$$\sigma(n)-2\sigma(n-1)-2\sigma(n-2)+3\sigma(n-3)+3\sigma(n-4)+3\sigma(n-5)$$
$$-4\sigma(n-6)-4\sigma(n-7)-4\sigma(n-8)-4\sigma(n-9)+5\sigma(n-10)+\dots$$
$$\dots+5\sigma(n-14)-\dots+(-1)^{r-1}r\sigma(1)=(-1)^s\tfrac{1}{6}(s^3-s).$$

In this formula the first term $\sigma(n)$ has the coefficient unity, the next two terms have 2 as coefficient, the next three have 3, and so on. The letter s denotes what the coefficient of $\sigma(0)$ would be if the series were continued one term further. Thus s is equal to r, unless $r\sigma(1)$ is the last term of the group of r terms having r as coefficient, in which case s is equal to $r+1$.

I now proceed to investigate a relation connecting together the actual divisors of all the numbers from unity to n, which includes the above formula as a particular case.

General Theorem relating to the actual Divisors of the First n Numbers.

§§ 45–50.

45. From § 28,

$$2(\sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+\&c.)\,\Sigma_1^\infty\,\sigma(\sin 2nx)\,q^n$$
$$=(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^3$$
$$+(\cos x+\cos 3x+\cos 5x-3\cos 7x)\,q^6-\&c.$$

Now, by multiplying the series

$$\sin x-q\sin 3x+q^3\sin 5x-\&c.$$

by

$$1+q+q^2+q^3+q^4+q^5+\&c.,$$

we obtain the series

$$\sin x-(\sin 3x-\sin x)(q+q^2)+(\sin 5x-\sin 3x+\sin x)(q^3+q^4+q^5)$$
$$-(\sin 7x-\sin 5x+\sin 3x-\sin x)(q^6+q^7+q^8+q^9)+\&c.,$$

which is equal to

$$\frac{1}{2\cos x}\left\{\sin 2x-(q+q^2)\sin 4x+(q^3+q^4+q^5)\sin 6x\right.$$
$$\left.-(q^6+\dots+q^9)\sin 8x+\&c.\right\};$$

and, by multiplying the series

$$(\cos x - \cos 3x)\, q - (\cos x + \cos 3x - 2\cos 5x)\, q^2 + \text{\&c.}$$

by $$1+q+q^2+q^3+q^4+q^5+\text{\&c.},$$

we obtain the series

$$\begin{aligned}&(\cos x - \cos 3x)(q+q^2) - (2\cos 3x - 2\cos 5x)(q^3+q^4+q^5)\\ &\quad + (\cos x - \cos 3x + 3\cos 5x - 3\cos 7x)(q^6+q^7+q^8+q^9)\\ &\quad - (2\cos 3x - 2\cos 5x + 4\cos 7x - 4\cos 9x)(q^{10}+\ldots+q^{14})\\ &\quad + \text{\&c.}\end{aligned}$$

Thus, by multiplying both sides of the above equation by

$$1+q+q^2+q^3+\text{\&c.},$$

we obtain the equation

$$\begin{aligned}&\{\sin 2x - (q+q^2)\sin 4x + (q^3+q^4+q^5)\sin 6x - \text{\&c.}\}\ \Sigma_1^\infty\, \sigma(\sin 2nx)\, q^n\\ &= \cos x\,\{(\cos x - \cos 3x)(q+q^2) - (2\cos 3x - 2\cos 5x)(q^3+q^4+q^5) - \text{\&c.}\},\end{aligned}$$

the right-hand member of which

$$\begin{aligned}&= \tfrac{1}{2}(1-\cos 4x)(q+q^2) - \tfrac{1}{2}(2\cos 2x - 2\cos 6x)(q^3+q^4+q^5)\\ &\qquad + \tfrac{1}{2}(1+2\cos 4x - 3\cos 8x)(q^6+q^7+q^8+q^9) - \text{\&c.}\end{aligned}$$

We thus find, by putting $\tfrac{1}{2}x$ for x,

$$\begin{aligned}&2\{\sin x - (q+q^2)\sin 2x + (q^3+q^4+q^5)\sin 3x - (q^6+\ldots+q^9)\sin 4x + \text{\&c.}\}\\ &\qquad\qquad \times \Sigma_1^\infty\, \sigma(\sin nx)\, q^n\\ &= (1-\cos 2x)(q+q^2) - (2\cos x - 2\cos 3x)(q^3+q^4+q^5)\\ &\qquad + (1+2\cos 2x - 3\cos 4x)(q^6+\ldots+q^9)\\ &\qquad - \text{\&c.},\end{aligned}$$

the general term on the right-hand side being

$$\{1+2\cos 2x+2\cos 4x+\ldots+2\cos(r-2)x-(r-1)\cos rx\}$$
$$\times\,(q^{\frac{1}{2}r(r-1)}+\ldots+q^{\frac{1}{2}r(r+1)-1}),$$

or $$-\{2\cos x+2\cos 3x+\ldots+2\cos(r-2)x-(r-1)\cos rx\}$$
$$\times\,(q^{\frac{1}{2}r(r-1)}+\ldots+q^{\frac{1}{2}r(r+1)-1}),$$

according as r is even or uneven.

46. The coefficient of q^n on the left-hand side of this equation is

$$\Sigma_n \left\{\cos(d-1)x-\cos(d+1)x\right\}$$

$$-(\Sigma_{n-1}+\Sigma_{n-2})\left\{\cos(d-2)x-\cos(d+2)x\right\}$$

$$+(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\left\{\cos(d-3)x-\cos(d+3)x\right\}$$

$$-\&c.;$$

so that, using the notation of § 29, in which $[a]$ denotes the absolute magnitude of a, irrespective of sign, we find that the numbers given by the formula,

$$G_n\left(\begin{matrix} d+1 \\ -[d-1]\end{matrix}\right)-(G_{n-1}+G_{n-2})\left(\begin{matrix} d+2 \\ -[d-2]\end{matrix}\right)$$

$$+(G_{n-3}+G_{n-4}+G_{n-5})\left(\begin{matrix} d+3 \\ -[d-3]\end{matrix}\right)-\&c.,$$

all cancel one another with the exception of

-0, two (-2)'s, two (-4)'s, ..., two $\{-(p-2)\}$'s, and $(p-1)$ p's,

if p be even, and

two 1's, two 3's, ..., two $(p-2)$'s and $(p-1)(-p)$'s,

if p be uneven, where $\frac{1}{2}p(p+1)$ is the triangular number next superior to n.* Zeros are to be retained and treated in the same manner as other numbers.

---

* It may be convenient to distinguish between the meanings of the letters h (§ 15), k (§ 16), l (§ 24), and p, which are used in this paper in connexion with the triangular numbers which are adjacent to, or equal, to n. The number $\frac{1}{2}h(h+1)$ is the triangular number next inferior to, n and $\frac{1}{2}p(p+1)$ is the triangular number next superior to n, whether n is a triangular number or not; $\frac{1}{2}k(k+1)$ is the triangular number next inferior to n, if n is not a triangular number, and is equal to n if n is a triangular number; $\frac{1}{2}l(l+1)$ is the triangular number next superior to n, if n is not a triangular number, and is equal to n if n is a triangular number. Thus, if n is not a triangular number,

$$h=k \text{ and } l=p=h+1=k+1,$$

and, if n is a triangular number, $\frac{1}{2}g(g+1)$,

$$h=g-1, \quad k=l=g, \quad p=g+1.$$

In the notation of §§ 5 and 29, the above formula may be written

$$\left\{\begin{matrix} \alpha+1, & \beta+1, & \dots \\ -(\alpha-1), & -(\beta-1), & \dots \end{matrix}\right\}$$

$$-\left\{\begin{matrix} \alpha_1-2, & \beta_1+2, & \dots; & \alpha_2+2, & \beta_2+2, & \dots \\ -[\alpha_1-2], & -[\beta_1-2], & \dots; & -[\alpha_2-2], & -[\beta_2-2], & \dots \end{matrix}\right\}$$

$$+\left\{\begin{matrix} \alpha_3+3, & \beta_3+3, & \dots; & \alpha_4+3, & \beta_4+3, & \dots; \\ -[\alpha_3-3], & -[\beta_3-3], & \dots; & -[\alpha_4-3], & -[\beta_4-3], & \dots; \\ & & & \alpha_5+3, & \beta_5+3, & \dots \\ & & & -[\alpha_5-3], & -[\beta_5-3], & \dots \end{matrix}\right\} - \&c.$$

47. As an example, putting $n=6$, so that $p=4$, the theorem asserts that

$$\left\{\begin{matrix} 2, & 3, & 4, & 7 \\ -0, & -1, & -2, & -5 \end{matrix}\right\} - \left\{\begin{matrix} 3, & 7; & 3, & 4, & 6 \\ -1, & -3; & -1, & -0, & -2 \end{matrix}\right\}$$

$$+\left\{\begin{matrix} 4, & 6; & 4, & 5; & 4 \\ -2, & -0; & -2, & -1; & -2 \end{matrix}\right\}$$

all cancel each other excepting only

$$-0, \; -2, \; -2, \; 4, \; 4, \; 4.$$

Again, putting $n=7$, so that $p=4$ as before, the theorem asserts that

$$\left\{\begin{matrix} 2, & 8 \\ -0, & -6 \end{matrix}\right\} - \left\{\begin{matrix} 3, & 4, & 5, & 8; & 3, & 7 \\ -1, & -0, & -1, & -4; & -1, & -3 \end{matrix}\right\}$$

$$+\left\{\begin{matrix} 4, & 5, & 7; & 4, & 6; & 4, & 5 \\ -2, & -1, & -1; & -2, & -0; & -2, & -1 \end{matrix}\right\} - \left\{\begin{matrix} 5 \\ -3 \end{matrix}\right\}$$

all cancel each other excepting only

$$-0, \; -2, \; -2, \; 4, \; 4, \; 4.$$

48. The theorem may be most conveniently exhibited in a form similar to that by which the theorem of § 29 was expressed in § 30. We first write down in the central line all the divisors of the numbers n, $n-1$, ..., 3, 2, 1. We then add 1 to each of the divisors of the first number n, writing the numbers so obtained above them; we add 2 to the divisors of the next two numbers, writing the numbers below; we add 3 to the next three sets of divisors, writing the numbers

above; and so on. Thus, in the case of $n=6$, we first form the scheme

$$\begin{array}{c|cc|ccc} 2,3,4,7 & & & 4,6, & 4,5, & 4, \\ (1,2,3,6) & (1,5), & (1,2,4) & (1,3), & (1,2), & (1). \\ & 3,7, & 3,4,6 & & & \end{array}$$

To complete the system of numbers, we subtract 1 from each of the divisors of n, writing the numbers below; we subtract 2 from the next two sets, writing the numbers above; 3 from the next three sets, writing the numbers below; and so on: in all cases attending only to the absolute values of the numbers (*i.e.*, ignoring negative signs). Thus, in the above example, we obtain the completed scheme

$$\begin{array}{c|cc|ccc} 2,3,4,7 & 1,3, & 1,0,2 & 4,6, & 4,5, & 4, \\ (1,2,3,6) & (1,5), & (1,2,4) & (1,3), & (1,2), & (1), \\ 0,1,2,5 & 3,7, & 3,4,6 & 2,0, & 2,1, & 2. \end{array}$$

The theorem asserts that, in general, if $\tfrac{1}{2}p\,(p+1)$ be the triangular number next superior to n, then, after cancelling the numbers which occur both in the upper and lower lines (ignoring the middle line which contains the divisors themselves), there will be left remaining, if p be even, $(p-1)$ p's in the upper line, and one 0, two 2's, two 4's, ..., and two $(p-2)$'s in the lower line, and, if p be uneven, two 1's, two 3's, ..., two $(p-2)$'s in the upper line, and $(p-1)\,p$'s in the lower line.

Thus, in the above scheme, for which $p=4$, there remain uncancelled three 4's in the upper line, and 0, 2, 2 in the lower line.

40. As additional examples, let $n=8$ and 9, the value of p being therefore 4 in each case. The schemes are

$$\begin{array}{c|cc|ccc|cc} 2,3,5,9 & 1,5, & 1,0,1,4 & 4,8, & 4,5,7, & 4,6 & 3,2, & 3 \\ (1,2,4,8) & (1,7), & (1,2,3,6) & (1,5), & (1,2,4), & (1,3) & (1,2), & (1) \\ 0,1,3,7 & 3,9, & 3,4,5,8 & 2,2, & 2,1,1, & 2,0 & 5,6, & 5 \end{array}$$

and

$$\begin{array}{c|cc|ccc|} 2,4,10 & 1,0,2,6, & 1,5 & 4,5,6,9, & 4,8, & 4,5,7 \\ (1,3,9) & (1,2,4,8), & (1,7) & (1,2,3,6), & (1,5), & (1,2,4) \\ 0,2,8 & 3,4,6,10, & 3,9 & 2,1,0,3, & 2,2, & 2,1,1 \end{array}$$

$$\begin{array}{ccc} 3,1, & 3,2, & 3 \\ (1,3), & (1,2), & (1) \\ 5,7, & 5,6, & 5. \end{array}$$

In each of these schemes, after the cancelling, three 4's are left in the upper line, and 0, 2, 2 in the lower.

For $n = 10$, the scheme is

$$\begin{array}{c|cc|ccc|}
2, 3, 6, 11 & 1, 1, 7, & 1, 0, 2, 6 & 4, 10, & 4, 5, 6, 9, & 4, 8 \\
(1, 2, 5, 10) & (1, 3, 9), & (1, 2, 4, 8) & (1, 7), & (1, 2, 3, 6), & (1, 5) \\
0, 1, 4, 9 & 3, 5, 11, & 3, 4, 6, 11 & 2, 4, & 2, 1, 0, 3, & 2, 2
\end{array}$$

$$\begin{array}{cccc}
3, 2, 0, & 3, 1, & 3, 2, & 3 \\
(1, 2, 4), & (1, 3), & (1, 2), & (1) \\
5, 6, 8, & 5, 7, & 5, 6, & 5\,;
\end{array}$$

and, after the cancelling, 1, 1, 3, 3 are left in the upper line, and four 5's in the lower line, as should be the case since $p = 5$.

50. In forming the schemes we divide the numbers beginning with n and proceeding downwards into sets of one, two, three, &c. The last set may be incomplete, as in the cases of $n = 8$ and 9 (when it consists of 2 and 3 numbers respectively), or complete, as in the case of $n = 10$ (when it consists of the full number 4). In the former case (*i.e.*, when the set is incomplete) p is equal to the full number of numbers which would belong to the set; but when the set is complete, p is equal to the number of terms belonging to the set, increased by unity. Thus, for $n = 8$ or 9, $p = 4$; but for $n = 10$, $p = 5$.

The preceding theorem (§§ 46–48) seems to me to be the most interesting of the results contained in this paper, as it connects together in so simple a manner all the actual divisors of the numbers from 1 to n.

Numerical Theorems relating to the Sums of Powers of the Divisors of the First n Numbers. §§ 51–56.

51. By proceeding as in § 31, we see that, if ψ be any even function,

$$\Sigma_n \{\psi(d+1) - \psi(d-1)\} - (\Sigma_{n-1} + \Sigma_{n-2}) \{\psi(d+2) - \psi(d-2)\}$$
$$+ (\Sigma_{n-3} + \Sigma_{n-4} + \Sigma_{n-5}) \{\psi(d+3) - \psi(d-3)\} - \&c.$$

is equal to

$$-\ \psi(0) - 2\psi(2) - 2\psi(4) - \ldots - 2\psi(p-2) + (p-1)\,\psi(p),$$

or $$2\psi(1) + 2\psi(3) + 2\psi(5) + \ldots + 2\psi(p-2) - (p-1)\,\psi(p),$$

according as p is even or uneven, $\frac{1}{2}p\,(p+1)$ being the triangular number next superior to n.

We may also define p as the number of terms in the complete set or group of terms to which Σ_0 would belong, if the series were continued one term further (so as to include Σ_0). It is evident that Σ_0 will belong to the same set as Σ_1 except when Σ_1 is the last term of its set. As the series is supposed to be continued to Σ_0 merely to determine p, the value zero is, of course, to be assigned to this term.

52. As a particular case, putting

$$\psi\,(d+1) = (d+1)^2,$$

we obtain the theorem

$$\Sigma_n 4d - (\Sigma_{n-1} + \Sigma_{n-2})\, 8d + (\Sigma_{n-3} + \Sigma_{n-4} + \Sigma_{n-5})\, 24d - \&c.$$

$$= -2\,\{2^2+4^2+6^2+\ldots+(p-2)^2\} + (p-1)\,p^2,$$

or

$$2\,\{1^2+3^2+5^2+\ldots+(p-2)^2\} - (p-1)\,p^2,$$

according as p is even or uneven.

The right-hand member of this equation may be written

$$-2\,(2^2+4^2+6^2+\ldots+p^2)+p^3+p^2,$$

or

$$2\,(1^2+3^2+5^2+\ldots+p^2)-p^3-p^2,$$

according as p is even or uneven.

Now, p being even,

$$2^2+4^2+6^2+\ldots+p^2 = \tfrac{1}{6}p^3+\tfrac{1}{2}p^2+\tfrac{1}{3}p,$$

and, p being uneven,

$$1^2+3^2+5^2+\ldots+p^2 = \tfrac{1}{6}p^3+\tfrac{1}{2}p^2+\tfrac{1}{3}p\,;$$

so that, whether p be even or uneven, we find

$$\Sigma_n d - 2\,(\Sigma_{n-1}+\Sigma_{n-2})\,d + 3\,(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\,d - \&c. = (-1)^p\,\tfrac{1}{6}\,(p^3-p),$$

which is the same as the formula quoted in § 44.

53. In general, putting

$$\psi\,(d+1) = (d+1)^m,$$

m being any uneven number, the left-hand member of the equation becomes

$$\begin{aligned}2\,(m+1)^{(1)}\Big[\sigma_m(n)-2\,\{\sigma_m(n-1)+\sigma_m\,(n-2)\}&\\ +3\,\{\sigma_m\,(n-3)+\sigma_m\,(n-4)+\sigma_m\,(n-5)\}&\\ -4\,\{\text{next four}\}+5\,\{\ldots\}-\&\text{c.}\Big]&\\ +2\,\frac{(m+1)^{(3)}}{3!}\Big[\sigma_{m-2}\,(n)-2^3\,\{\sigma_{m-2}\,(n-1)+\sigma_{m-2}\,(n-2)\}&\\ +3^3\,\{\text{next three}\}-4^3\,\{\ldots\}+\&\text{c.}\Big]&\\ +2\,\frac{(m+1)^{(5)}}{5!}\Big[\sigma_{m-4}\,(n)-2^5\,\{\sigma_{m-4}\,(n-1)+\sigma_{m-4}(n-2)\}+3^5\{\ldots\}-\&\text{c.}\Big]&\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots&\\ +2\,(m+1)^{(1)}\Big[\sigma\,(n)-2^m\,\{\sigma\,(n-1)+\sigma\,(n-2)\}+3^m\,\{\ldots\}-\&\text{c.}\Big],&\end{aligned}$$

where, as before, $m^{(r)}$ denotes the factorial $m\,(m-1)\ldots(m-r+1)$.

The right-hand member of the equation is

$$-2\,\{2^{m+1}+4^{m+1}+6^{m+1}+\ldots+p^{m+1}\}+p^{m+2}+p^{m+1},$$

or

$$2\,\{1^{m+1}+3^{m+1}+5^{m+1}+\ldots+p^{m+1}\}-p^{m+2}-p^{m+1},$$

according as p is even or uneven.

Now it can be shown* that the expression

$$\tfrac{1}{2}\,\frac{p^{m+2}}{m+2}+\tfrac{1}{2}p^{m+1}+(m+1)^{(1)}\,B_1p^m-2^2\,\frac{(m+1)^{(3)}}{3\,!}\,\frac{B_2}{2}\,p^{m-2}$$
$$+2^4\,\frac{(m+1)^{(5)}}{5\,!}\,\frac{B_3}{3}\,p^{m-4}-\ldots\pm 2^{m-1}\,(m+1)^{(1)}\,\frac{B_{\frac{1}{2}(m+1)}}{\tfrac{1}{2}\,(m+1)}\,p$$

represents the value of the series

$$2^{m+1}+4^{m+1}+6^{m+1}+\ldots+p^{m+1},$$

if p be even, and the value of the series

$$1^{m+1}+3^{m+1}+5^{m+1}+\ldots+p^{m+1},$$

if p be uneven.

---

* The proof of this theorem forms the subject of a paper, "Note on the Sums of Even Powers of Even and Uneven Numbers" (*Messenger*, Vol. xx., pp. 172–176), which contains the investigations to which I was led in obtaining the values of the series which form the right-hand member of the above equation.

We find, therefore, that, both for even and uneven values of p, the right-hand member of the equation is equal to

$$(-1)^p \left\{ \frac{m+1}{m+2} p^{m+2} - 2\,(m+1)^{(1)} B_1 p^m + 2^3 \frac{(m+1)^{(3)}}{3!} \frac{B_2}{2} p^{m-2} \right.$$
$$\left. - 2^5 \frac{(m+1)^{(5)}}{5!} \frac{B_3}{3} p^{m-4} + \text{\&c.} \right\}.$$

54. By dividing throughout by $2\,(m+1)$, we thus obtain the theorem

$$\sigma_m(n) - 2\,\{\sigma_m(n-1) + \sigma_m(n-2)\}$$
$$+ 3\,\{\sigma_m(n-3) + \sigma_m(n-4) + \sigma_m(n-5)\} - \text{\&c.}$$
$$+ \frac{m^{(2)}}{3!} \left[\sigma_{m-2}(n) - 2^3\,\{\sigma_{m-2}(n-1) + \sigma_{m-2}(n-2)\} + 3^3\{\dots\} - \text{\&c.}\right]$$
$$+ \frac{m^{(4)}}{5!} \left[\sigma_{m-4}(n) - 2^5\,\{\sigma_{m-4}(n-1) + \sigma_{m-4}(n-2)\} + 3^5\,\{\dots\} - \text{\&c.}\right]$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ \frac{m^{(2)}}{3!} \left[\sigma_3(n) - 2^{m-2}\,\{\sigma_3(n-1) + \sigma_3(n-2)\} + 3^{m-2}\,\{\dots\} - \text{\&c.}\right]$$
$$+ \sigma(n) - 2^m\,\{\sigma(n-1) + \sigma(n-2)\}$$
$$+ 3^m\,\{\sigma(n-3) + \sigma(n-4) + \sigma(n-5)\} - \text{\&c.}$$
$$= (-1)^p \left\{ \frac{p^{m+2}}{2\,(m+2)} - B_1 p^m + 2^2 \frac{m^{(2)}}{3!} \frac{B_2}{2} p^{m-2} - 2^4 \frac{m^{(4)}}{5!} \frac{B_3}{3} p^{m-4} + \dots \right.$$
$$\left. \dots \pm 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)} p \right\}.$$

55. As in § 41, we may represent all the terms on the right-hand side of the equation, with the exception of the first, by assigning to $\sigma(0)$, $\sigma_3(0)$, ... , $\sigma_m(0)$ numerical values which are independent of both n and m. These values are

$$\sigma(0) = -B_1, \quad \sigma_3(0) = 2^2 \frac{B_2}{2}, \quad \sigma_5(0) = -2^4 \frac{B_3}{3}, \dots,$$

$$\sigma_{2r-1}(0) = (-1)^r\, 2^{2r-2} \frac{B_r}{r}, \dots, \sigma_m(0) = (-1)^{\frac{1}{2}(m+1)}\, 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}.$$

Assigning these values to $\sigma(0)$, $\sigma_3(0)$, ... , $\sigma_m(0)$, we may write the theorem in the form:

$$\begin{aligned}
&\sigma_m(n)-2\{\sigma_m(n-1)+\sigma_m(n-2)\}+3\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p\{\ldots+\sigma_m(0)\} \\
&+\frac{m^{(2)}}{3!}\Big[\sigma_{m-2}(n)-2^3\{\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)\} \\
&\qquad\qquad+3^3\{\ldots\}-\ldots+(-1)^{p-1}p^3\{\ldots+\sigma_{m-2}(0)\}\Big] \\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\
&+\frac{m^{(2)}}{3!}\Big[\sigma_3(n)-2^{m-2}\{\sigma(n-1)+\sigma(n-2)\} \\
&\qquad\qquad+3^{m-2}\{\ldots\}-\ldots+(-1)^{p-1}p^{m-2}\{\ldots+\sigma_3(0)\}\Big] \\
&+\sigma(n)-2^m\{\sigma(n-1)+\sigma(n-2)\}+3^m\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p^m\{\ldots+\sigma(0)\} \\
&\qquad=(-1)^p\frac{p^{m+2}}{2(m+2)}.^*
\end{aligned}$$

56. As examples of this general theorem, putting $m=3$ and 5, we have

$$\begin{aligned}
&\sigma_3(n)-2\{\sigma_3(n-1)+\sigma_3(n-2)\}+3\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p\{\ldots+\sigma_3(0)\} \\
&+\sigma(n)-2^3\{\sigma(n-1)+\sigma(n-2)\}+3^3\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p^3\{\ldots+\sigma(0)\} \\
&\qquad=(-1)^p\tfrac{1}{16}p^5,
\end{aligned}$$

where $\sigma(0)=-\frac{1}{6},\quad \sigma_0(0)=\frac{1}{15},$

and

$$\begin{aligned}
&\sigma_5(n)-2\{\sigma_5(n-1)+\sigma_5(n-2)\}+3\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p\{\ldots+\sigma_5(0)\} \\
&+\tfrac{10}{3}\Big[\sigma_3(n)-2^3\{\sigma_3(n-1)+\sigma_3(n-2)\}+3^3\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p^3\{\ldots+\sigma(0)\}\Big] \\
&+\sigma(n)-2^5\{\sigma(n-1)+\sigma(n-2)\}+3^5\{\ldots\}-\ldots \\
&\qquad\qquad\ldots+(-1)^{p-1}p^5\{\ldots+\sigma(0)\} \\
&\qquad=(-1)^p\tfrac{1}{14}p^7,
\end{aligned}$$

where $\sigma(0)=-\frac{1}{6},\quad \sigma_3(0)=\frac{1}{15},\quad \sigma_5(0)=-\frac{8}{63}.$

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[*] We may evidently dispense with this term by putting

$$\sigma(0)=\frac{p^2}{2(m+2)}-\tfrac{1}{6};$$

$\sigma_3(0)$, $\sigma_5(0)$, ..., retaining their constant values as above. We may regard p as defined by the fact that p^m is the coefficient of $\sigma(0)$.

Second Theorem connecting the actual Divisors of the First n Numbers.

§§ 57–60.

57. The series

$$\sin x-(\sin 3x-\sin x)(q+q^2)$$
$$+(\sin 5x-\sin 3x+\sin x)(q^3+q^4+q^5)-\&c.,$$

which was obtained (§ 45) by multiplying $\sin x-q\sin 3x+q^3\sin 5x-\&c.$ by $1+q+q^2+q^3+\&c.$, may be expressed in the form

$$\sin x\,\{1-2\cos 2x\,(q+q^2)+(1+2\cos 4x)(q^3+q^4+q^5)$$
$$-(2\cos 2x+2\cos 6x)(q^6+\ldots+q^9)+\&c.\},$$

and the series

$$(\cos x-\cos 3x)(q+q^2)-(2\cos 3x-2\cos 5x)(q^3+q^4+q^5)+\&c.,$$

which was obtained (§ 45) by multiplying

$$(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^3+\&c.$$

by
$$1+q+q^2+q^3+\&c.,$$

may be expressed in the form

$$2\sin x\,\{\sin 2x\,(q+q^2)-2\sin 4x\,(q^3+q^4+q^5)$$
$$+(\sin 2x+3\sin 6x)(q^6+\ldots+q^9)-\&c.\}.$$

Putting $\frac{1}{2}x$ for x, we thus obtain the equation

$$\{1-2\cos x\,(q+q^2)+(1+2\cos 2x)(q^3+q^4+q^5)$$
$$-(2\cos x+2\cos 3x)(q^6+\ldots+q^9)+(1+2\cos 2x+2\cos 4x)(q^{10}+\ldots+q^{14})$$
$$-(2\cos x+2\cos 3x+2\cos 5x)(q^{15}+\ldots+q^{20})$$
$$+(1+2\cos 2x+2\cos 4x+2\cos 6x)(q^{21}+\ldots+q^{27})-\&c.\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=\sin x\,(q+q^2)-2\sin 2x\,(q^3+q^4+q^5)+(\sin x+3\sin 3x)(q^6+\ldots+q^9)$$
$$-(2\sin 2x+4\sin 4x)(q^{10}+\ldots+q^{14})$$
$$+(\sin x+3\sin 3x+5\sin 5x)(q^{15}+\ldots+q^{20})-\&c.$$

58. The coefficient of q^n on the left-hand side is

$$\Sigma_n\sin dx-(\Sigma_{n-1}+\Sigma_{n-2})\,\{\sin\,(d\pm 1)\,x\}$$
$$+(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\,\{\sin dx+\sin\,(d\pm 2)\,x\}$$
$$-(\Sigma_{n-6}+\ldots+\Sigma_{n-9})\,\{\sin\,(d\pm 1)\,x+\sin\,(d\pm 3)\,x\}$$
$$+(\Sigma_{n-10}+\ldots+\Sigma_{n-14})\,\{\sin dx+\sin\,(d\pm 2)\,x+\sin\,(d\pm 4)\,x\}$$
$$-\&c.,$$

and we see, therefore, that the numbers given by the formula

$$\begin{aligned} &G_n(d)-(G_{n-1}+G_{n-2})(d\pm 1)+(G_{n-3}+G_{n-4}+G_{n-5})(d,\, d\pm 2) \\ &\qquad -(G_{n-6}+\ldots+G_{n-9})(d\pm 1,\, d\pm 3) \\ &\qquad +(G_{n-10}+\ldots+G_{n-14})(d,\, d\pm 2,\, d\pm 4)-\&\text{c.} \end{aligned}$$

all cancel each other, with the exception of

one 1, three 3's, five 5's, ..., $(p-1)(p-1)$'s, if p be even,

and

two (-2)'s, four (-4)'s, six (-6)'s,..., $(p-1)\{-(p-1)\}$'s, if p be uneven,

where $\frac{1}{2}p\,(p+1)$ is the triangular number next superior to n. Zeros are not to be taken account of.

59. As an example, putting $n=9$, the theorem asserts that the numbers

$$\{1, 3, 9\}-\left\{\begin{array}{ll} 2, 3, 5, 9, & 2, 8 \\ (1, 2, 4, 8), & (1, 7) \\ 0, 1, 3, 7, & 0, 6 \end{array}\right\}+\left\{\begin{array}{lll} 3, 4, 5, 8, & 3, 7, & 3, 4, 6 \\ 1, 2, 3, 6, & 1, 5, & 1, 2, 4 \\ -1, 0, 1, 4, & -1, 3, & -1, 0, 2 \end{array}\right\}$$

$$-\left\{\begin{array}{llll} 4, 6, & 4, & 5, & 4 \\ 2, 4, & 2, & 3, & 2 \\ (1, 3), & (1, & 2), & (1) \\ 0, 2, & 0, & 1, & 0 \\ -2, 0, & -2, & -1, & -2 \end{array}\right\}$$

all cancel each other, excepting only 1, 3, 3, 3. The divisors in round brackets are shown for convenience, but are not to be included among the numbers. Thus the numbers given by the formula are

$$\begin{array}{llll} & & & -4, -6, -4, -5, -4 \\ & & 3, 4, 5, 8, \;\; 3, 7, \;\; 3, 4, 6, & \\ & -2, -3, -5, -9, -2, -8, & & -2, -4, -2, -3, -2 \\ 1, 3, 9, & & 1, 2, 3, 6, \;\; 1, 5, \;\; 1, 2, 4, & \\ & 0, -1, -3, -7, \;\; 0, -6, & & 0, -2, \;\; 0, -1, \;\; 0 \\ & & -1, 0, 1, 4, -1, 3, -1, 0, 2, & \\ & & & 2, \;\; 0, \;\; 2, \;\; 1, \;\; 2 \end{array}$$

and it is easily seen that, after cancelling, 1, 3, 3, 3 are alone left. Putting $n = 10$, we form the scheme

$$\{1, 2, 5, 10\} - \left\{\begin{array}{ll} 2, 4, 10, & 2, 3, 5, 9 \\ (1, 3, 9), & (1, 2, 4, 8) \\ 0, 2, 8, & 0, 1, 3, 7 \end{array}\right\} + \left\{\begin{array}{lll} 3, 9, & 3, 4, 5, 8, & 3, 7 \\ 1, 7, & 1, 2, 3, 6, & 1, 5 \\ -1, 5, & -1, 0, 1, 4, & -1, 3 \end{array}\right\}$$

$$- \left\{\begin{array}{llllllll} 4, & 5, 7, & 4, 6, & 4, & 5, & 4 \\ 2, & 3, 5, & 2, 4, & 2, & 3, & 2 \\ (1, & 2, 4), & (1, 3), & (1, & 2), & (1) \\ 0, & 1, 3, & 0, 2, & 0, & 1, & 0 \\ -2, & -1, 1, & -2, 0, & -2, & -1, & -2 \end{array}\right\};$$

the numbers given by the formula are therefore

$$\begin{array}{r} -4, -5, -7, -4, -6, -4, -5, -4 \\ 3, 9, \quad 3, 4, 5, 8, \quad 3, 7, \qquad\qquad\qquad\qquad\qquad\qquad \\ -2, -4, -10, -2, -3, -5, -9, \qquad -2, -3, -5, -2, -4, -2, -3, -2 \\ 1, 2, 5, 10, \qquad\qquad\qquad\qquad 1, 7, \quad 1, 2, 3, 6, \quad 1, 5, \qquad\qquad\qquad\qquad\qquad\qquad \\ 0, -2, -8, \quad 0, -1, -3, -7, \qquad 0, -1, -3, \quad 0, -2, \quad 0, -1, \quad 0 \\ -1, 5, -1, 0, 1, 4, -1, 3, \qquad\qquad\qquad\qquad\qquad\qquad \\ 2, \quad 1, -1, \quad 2, \quad 0, \quad 2, \quad 1, \quad 2 \end{array}$$

which reduce to

$$-2, -2, -4, -4, -4, -4,$$

as should be the case, since $p = 5$.

Numerical Theorems relating to the sums of Powers of the First n Numbers. §§ 60–62.

60. From the theorem relating to the actual divisors it follows immediately that, if ϕ be any uneven function,

$$\begin{aligned} & \Sigma_n \phi(d) - (\Sigma_{n-1} + \Sigma_{n-2}) \{ \phi(d+1) + \phi(d-1) \} \\ & + (\Sigma_{n-3} + \Sigma_{n-4} + \Sigma_{n-5}) \{ \phi(d) + \phi(d+2) + \phi(d-2) \} \\ & - (\Sigma_{n-6} + \ldots + \Sigma_{n-9}) \{ \phi(d+1) + \phi(d-1) + \phi(d+3) + \phi(d-3) \} \\ & + \&c. \end{aligned}$$

is equal to

$$\phi(1) + 3\phi(3) + 5\phi(5) + \ldots + (p-1)\phi(p-1), \text{ if } p \text{ be even.}$$

and to

$$-2\phi(2)-4\phi(4)-6\phi(6)-\ldots-(p-1)\,\phi(p-1), \text{ if } p \text{ be uneven,}$$

where, as before, $\frac{1}{2}p\,(p+1)$ is the triangular number next superior to n.

61. By putting $\phi(d) = d$ we obtain the original formula quoted in § 44, which was also obtained in § 52 by putting $\psi(d+1) = (d+1)^2$.

If we put $\phi(d) = d^m$, m being uneven, we obtain the general theorem

$$\sigma_m(n)-2\left\{\sigma_m(n-1)+\sigma_m(n-2)\right\}$$
$$+3\left\{\sigma_m(n-3)+\sigma_m(n-4)+\sigma_m(n-5)\right\}-4\left\{\text{next four}\right\}+\&\text{c.}$$
$$= 2\,\frac{m^{(2)}}{2\,!}\Big[\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)-2^2\left\{\text{next three}\right\}$$
$$+(1^2+3^2)\left\{\text{next four}\right\}-(2^2+4^2)\left\{\text{next five}\right\}+\&\text{c.}\Big]$$
$$+2\,\frac{m^{(4)}}{4\,!}\Big[\sigma_{m-4}(n-1)+\sigma_{m-4}(n-2)-2^4\left\{\text{next three}\right\}$$
$$+(1^4+3^4)\left\{\text{next four}\right\}-\&\text{c.}\Big]$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+2m\Big[\sigma(n-1)+\sigma(n-2)-2^{m-1}\left\{\text{next three}\right\}$$
$$+(1^{m+1}+3^{m-1})\left\{\text{next four}\right\}-\&\text{c.}\Big]$$
$$+\begin{cases} \quad 1^{m+1}+3^{m+1}+5^{m+1}+\ldots+(p-1)^{m+1}, \text{ if } p \text{ be even,} \\ \text{or} \quad -2^{m+1}-4^{m+1}-6^{m+1}-\ldots-(p-1)^{m+1}, \text{ if } p \text{ be uneven,} \end{cases}$$

p being defined as in the preceding section.

62. The coefficients in the series on the right-hand side are of the forms $\quad 1^{2r}+3^{2r}+5^{2r}+\ldots \quad$ and $\quad 2^{2r}+4^{2r}+6^{2r}+\ldots$

alternately, according as the number of terms in the group is even or uneven. This apparent alternation of form disappears if we express each coefficient as a function of the number of terms in the group. For (by § 53) the function

$$\tfrac{1}{2}\,\frac{x^{2r+1}}{2r+1}+\tfrac{1}{2}x^{2r}+(2r)^{(1)}B_1\,x^{2r-1}-2^2\,\frac{(2r)^{(3)}}{3\,!}\,\frac{B_2}{2}\,x^{2r-3}+2^4\,\frac{(2r)^{(5)}}{5\,!}\,\frac{B_3}{3}\,x^{2r-5}-\ldots$$
$$\ldots\pm 2^{2r-2}\,(2r)^{(1)}\,\frac{B_r}{r}\,x$$

is equal to

$$2^{2r}+4^{2r}+6^{2r}+\ldots+x^{2r} \quad \text{or} \quad 1^{2r}+3^{2r}+5^{2r}+\ldots+x^{2r},$$

according as r is even or uneven, so that the coefficient of the s^{th} group of terms in the r^{th} right-hand series, *i.e.*, the coefficient of

$$\sigma_{m-2r}(n-t)+\sigma_{m-2r}(n-t-1)+\ldots+\sigma_{m-2r}(n-t-s),$$

where t denotes $\frac{1}{2}s(s+1)$, is equal to

$$(-1)^{s-1}\left\{\tfrac{1}{2}\frac{s^{2r+1}}{2r+1}+\tfrac{1}{2}s^{2r}+(2r)^{(1)}B_1\,s^{2r-1}-2^3\frac{(2r)^{(3)}}{3\,!}\frac{B_2}{2}s^{2r-3}+\&c.\right\},$$

and the last term is equal to

$$(-1)^{l-1}\left\{\tfrac{1}{2}\frac{l^{m+2}}{m+2}+\tfrac{1}{2}l^{m+1}+(m+1)^{(1)}B_1\,l^{m}-2^3\frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2}l^{m-2}+\&c.\right\}.$$

The letter s may be regarded as denoting either the number of the group, or the number of terms in the group diminished by unity; and $l=p-1$, so that $\frac{1}{2}l(l+1)$ is the triangular number next inferior to n or equal to n, according as n is not, or is, a triangular number.

Other Formulæ relating to the actual Divisors. §§ 63–67.

63. The algebraical formulæ from which the four theorems relating to the actual divisors (§§ 3, 29, 46, 58) have been deduced are

(i.)

$$\{1-(1+2\cos x)\,q+(1+2\cos x+2\cos 2x)\,q^3-\&c.\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=\sin x\,.\,q-(\sin x+2\sin 2x)\,q^3+(\sin x+2\sin 2x+3\sin 3x)\,q^6-\&c.\ (\S\ 27.)$$

(ii.)

$$2\,\{\sin x-q\sin 3x+q^2\sin 5x-q^6\sin 7x+\&c.\}\,\Sigma_1^\infty\,\sigma\,(\sin 2nx)\,q^n$$
$$=(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^3$$
$$+(\cos x+\cos 3x+\cos 5x-3\cos 7x)\,q^6-\&c.\quad(\S\ 28.)$$

(iii.)

$$2\,\{\sin x-(q+q^2)\sin 2x+(q^3+q^4+q^5)\sin 3x-\&c.\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=(1-\cos 2x)(q+q^2)-(2\cos x-2\cos 3x)(q^3+q^4+q^5)$$
$$+(1+2\cos 2x-3\cos 4x)(q^6+\ldots+q^9)$$
$$-(2\cos x+2\cos 3x-4\cos 5x)(q^{10}+\ldots+q^{14})+\&c.\quad(\S\ 45.)$$

(iv.)

$$\{1-2\cos x\,(q+q^2)+(1+2\cos 2x)(q^3+q^4+q^5)$$
$$-(2\cos x+2\cos 3x)(q^6+\ldots+q^9)$$
$$+(1+2\cos 2x+2\cos 4x)(q^{10}+\ldots+q^{14})-\&c.\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$

$=\sin x\,(q+q^2)-2\sin 2x\,(q^3+q^4+q^5)+(\sin$ [illegible] $3x)(q^6+\ldots+q^9)$

$-(2\sin 2x+4$ [illegible] $+q^{14})+$ [illegible]$)$

The second of these formulæ is derivable from the first by multiplying by $\sin \frac{1}{2}x$ and replacing x by $2x$, and the third from the fourth by multiplying by $\sin x$.

64. We may obtain other formulæ relating to the actual divisors by multiplying these equations by $\sin x$ or $\cos x$, or other trigonometrical expressions, and equating coefficients.

Thus, for example, multiplying (ii.) by $\sin x$ and replacing x by $\frac{1}{2}x$, we find that

$$2\left\{1-\cos x-(\cos x-\cos 2x)\,q+(\cos 2x-\cos 3x)\,q^3 \right.$$
$$\left. -(\cos 3x-\cos 4x)\,q^6+\&c.\right\} \Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=(2\sin x-\sin 2x)\,q-(3\sin 2x-2\sin 3x)\,q^3+(4\sin 3x-3\sin 4x)\,q^6-\&c.,$$

the general term on the right-hand side being

$$(-1)^{g-1}\left\{(g+1)\sin gx-g\sin\,(g+1)\,x\right\}\,q^{\frac{1}{2}g(g+1)}.$$

65. Equating the coefficients of q^n, we see that the numbers given by the formula

$$G_n\left\{d,\,d,\,-(d\pm1)\right\}-G_{n-1}\left\{d\pm1,\,-(d\pm2)\right\}+G_{n-3}\left\{d\pm2,\,-(d\pm3)\right\}$$
$$-G_{n-6}\left\{d\pm3,\,-(d\pm4)\right\}+\&c.$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g\,(g+1)$, in which case

$$(g+1)(-g)\text{'s} \quad\text{and}\quad g\,(g+1)\text{'s},$$

or

$$(g+1)\,g\text{'s} \quad\text{and}\quad g\left\{-(g+1)\right\}\text{'s},$$

remain uncancelled, according as g is even or uneven. Zeros are not to be taken account of.

As an example, putting $n = 6$, so that $g = 3$, the formula gives the numbers

$$\left\{\begin{array}{rrrr} 1, & 2, & 3, & 6 \\ 1, & 2, & 3, & 6 \\ -2, & -3, & -4, & -7 \\ 0, & -1, & -2, & -5 \end{array}\right\} - \left\{\begin{array}{rr} 0, & 4 \\ 2, & 6 \\ -3, & -7 \\ 1, & -3 \end{array}\right\} + \left\{\begin{array}{rr} -1, & 1 \\ 3, & 5 \\ -4, & -6 \\ 2, & 0 \end{array}\right\},$$

which cancel each other, excepting only

$$3,\ 3,\ 3,\ 3,\ -4,\ -4,\ -4.$$

Again, putting $n = 10$, so that $g = 4$, the formula gives

$$\left\{\begin{array}{rrrr} 1, & 2, & 5, & 10 \\ 1, & 2, & 5, & 10 \\ -2, & -3, & -6, & -11 \\ 0, & -1, & -4, & -9 \end{array}\right\} - \left\{\begin{array}{rrr} 0, & 2, & 8 \\ 2, & 4, & 10 \\ -3, & -5, & -11 \\ 1, & -1, & -7 \end{array}\right\} + \left\{\begin{array}{rr} -1, & 5 \\ 3, & 9 \\ -4, & -10 \\ 2, & -4 \end{array}\right\}$$

$$- \left\{\begin{array}{rrr} -2, & -1, & 1 \\ 4, & 5, & 7 \\ -5, & -6, & -8 \\ 3, & 2, & 0 \end{array}\right\},$$

which cancel each other, excepting only

$$-4,\ -4,\ -4,\ -4,\ -4,\ 5,\ 5,\ 5,\ 5.$$

This theorem is noticeable on account of the curious form of the group of numbers which is left uncancelled, $(g+1)$ occurring g times, and g occurring $(g+1)$ times, with opposite signs.

66. By multiplying (ii.) by $\cos x$, and replacing x by $\tfrac{1}{2}x$, we find that

$$2\,\{\sin x - (\sin x + \sin 2x)\,q + (\sin 2x + \sin 3x)\,q^3$$
$$-(\sin 3x + \sin 4x)\,q^6 + \&c.\}\,\Sigma_1^\infty \sigma\,(\sin nx)\,q^n$$

$$= (1-\cos 2x)\,q - (1 + 2\cos x - \cos 2x - 2\cos 3x)\,q^3$$
$$+ (1 + 2\cos x + 2\cos 2x - 2\cos 3x - 3\cos 4x)\,q^6 + \&c.,$$

the coefficient of $(-1)^{g-1}\,q^{\frac{1}{2}g(g+1)}$ in the series on the right-hand side being

$$1 + 2\cos x + 2\cos 2x + \dots + 2\cos (g-1)x - (g-1)\cos gx - g\cos (g+1)\,x.$$

It follows, therefore, that the numbers given by the formula

$$G_n\begin{pmatrix} d+1 \\ -[d-1] \end{pmatrix} - G_{n-1}\begin{pmatrix} d+1, & d+2 \\ -[d-1], & -[d-2] \end{pmatrix}$$
$$+ G_{n-3}\begin{pmatrix} d+2, & d+3 \\ -[d-2], & -[d-3] \end{pmatrix} - \&c.$$

all cancel each other, unless n is a triangular number $\tfrac{1}{2}g\,(g+1)$, in which case the numbers left uncancelled are

0, two 1's, two 2's, ..., two $(g$ [illegible] $)(-g)$'s, $g\,\{$ [illegible] if g is even, and these numbers with their signs reversed if g is uneven. Zeros are to be treated in the same manner as other numbers.

Thus, putting $n = 6$, so that $g = 3$, the formula gives

$$\left\{\begin{array}{cccc} 2, & 3, & 4, & 7 \\ -0, & -1, & -2, & -5 \end{array}\right\} - \left\{\begin{array}{cc} 3, & 7 \\ 2, & 6 \\ -0, & -4 \\ -1, & -3 \end{array}\right\} + \left\{\begin{array}{cc} 4, & 6 \\ 3, & 5 \\ -1, & -1 \\ -2, & -0 \end{array}\right\},$$

in which the uncancelled numbers are

$$-0,\ -1,\ -1,\ -2,\ -2,\ 3,\ 3,\ 4,\ 4,\ 4.$$

Again, putting $n = 10$, so that $g = 4$, the formula gives

$$\left\{\begin{array}{cccc} 2, & 3, & 6, & 11 \\ -0, & -1, & -4, & -\ 9 \end{array}\right\} - \left\{\begin{array}{ccc} 3, & 5, & 11 \\ 2, & 4, & 10 \\ -0, & -2, & -\ 8 \\ -1, & -1, & -\ 7 \end{array}\right\} + \left\{\begin{array}{cc} 4, & 10 \\ 3, & 9 \\ -1, & -\ 5 \\ -2, & -\ 4 \end{array}\right\}$$

$$-\left\{\begin{array}{ccc} 5, & 6, & 8 \\ 4, & 5, & 7 \\ -2, & -1, & -1 \\ -3, & -2, & -0 \end{array}\right\},$$

in which the uncancelled numbers are

$$0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ -4,\ -4,\ -4,\ -5,\ -5,\ -5,\ -5.$$

67. As another example, multiplying (iii.) by $\cos x$, we find that

$$\begin{aligned} 2\,\{&\sin 2x - (\sin x + \sin 3x)(q+q^2) + (\sin 2x + \sin 4x)(q^3+q^4+q^5) \\ &- (\sin 3x + \sin 5x)(q^6 + \ldots + q^9) \\ &+ (\sin 4x + \sin 6x)(q^{10} + \ldots + q^{14}) - \&c.\}\ \Sigma_1^\infty\, \sigma\, (\sin nx) \\ = {}& (\cos x - \cos 3x)(q+q^2) - (2 - 2\cos 4x)(q^3+q^4+q^5) \\ &+ (4\cos x - \cos 3x - 3\cos 5x)(q^6 + \ldots + q^9) \\ &- (2 + 4\cos 2x - 2\cos 4x - 4\cos 6x)(q^{10} + \ldots + q^{14}) + \&c., \end{aligned}$$

the coefficient of the group of terms, on the right-hand side, beginning with $q^{\frac{1}{2}r(r-1)}$, being

$$-\left\{2+4\cos 2x+4\cos 4x+\ldots+4\cos (r-2)\,x\right.$$
$$\left.-(r-2)\cos rx-r\cos (r-2)\,x\right\},$$

or $4\cos x+4\cos 3x+\ldots+4\cos (r-2)\,x-(r-2)\cos rx-r\cos (r-2)\,x$,

according as r is even or uneven.

It follows, therefore, that the numbers given by the formula

$$G_n\begin{pmatrix} d+2 \\ -(d-2) \end{pmatrix}-(G_{n-1}+G_{n-2})\begin{pmatrix} d+1, & d+3 \\ -(d-1), & -(d-3) \end{pmatrix}$$
$$+(G_{n-3}+G_{n-4}+G_{n-5})\begin{pmatrix} d+2, & d+4 \\ -(d-2), & -(d-4) \end{pmatrix}$$
$$-(G_{n-6}+G_{n-7}+G_{n-8}+G_{n-9})\begin{pmatrix} d+3, & d+5 \\ -(d-3), & -(d-5) \end{pmatrix}+\&c.$$

all cancel each other, excepting

two 0's, four 2's, ... four $(k-2)$'s, $(k-2)(-k)$'s, $k\{-(k+2)\}$'s

when k is even, and

four (-1)'s, four (-3)'s, ... four $\{-(k-2)\}$'s, $(k-2)\,k$'s, $k(k+2)$'s

when k is uneven, $\frac{1}{2}k\,(k+1)$ being the triangular number which is next inferior to n, or equal to n, according as n is not, or is, a triangular number. Zeros are to be treated in the same manner as other numbers.

Thus, putting $n=6$, so that $k=3$, the theorem asserts that the numbers

$$\begin{Bmatrix} 3, & 4, & 5, & 8 \\ -1, & -0, & -1, & -4 \end{Bmatrix}-\begin{Bmatrix} 4, & 8, & 4, & 5, & 7 \\ 2, & 6, & 2, & 3, & 5 \\ -0, & -4, & -0, & -1, & -3 \\ -2, & -2, & -2, & -1, & -1 \end{Bmatrix}$$
$$+\begin{Bmatrix} 5, & 7, & 5, & 6, & 5 \\ 3, & 5, & 3, & 4, & 3 \\ 1, & 1, & -1, & -0, & -1 \\ 3, & 1, & -3, & -2, & -3 \end{Bmatrix}$$

all cancel each other, with the exception of

1, 1, 1, 1, 3, 5, 5, 5

If $n = 10$, so that $k = 4$, the numbers are

$$\left\{\begin{matrix} 3, & 4, & 7, & 12 \\ -1, & -0, & -3, & -8 \end{matrix}\right\} - \left\{\begin{matrix} 4, & 6, & 12, & 4, & 5, & 7, & 11 \\ 2, & 4, & 10, & 2, & 3, & 5, & 9 \\ -0, & -2, & -8, & -0, & -1, & -3, & -7 \\ -2, & -0, & -6, & -2, & -1, & -1, & -5 \end{matrix}\right\}$$

$$+ \left\{\begin{matrix} 5, & 11, & 5, & 6, & 7, & 10, & 5, & 9 \\ 3, & 9, & 3, & 4, & 5, & 8, & 3, & 7 \\ -1, & -5, & -1, & -0, & -1, & -4, & -1, & -3 \\ -3, & -3, & -3, & -2, & -1, & -2, & -3, & -1 \end{matrix}\right\}$$

$$- \left\{\begin{matrix} 6, & 7, & 9, & 6, & 8, & 6, & 7, & 6 \\ 4, & 5, & 7, & 4, & 6, & 4, & 5, & 4 \\ -2, & -1, & -1, & -2, & -0, & -2, & -1, & -2 \\ -4, & -3, & -1, & -4, & -2, & -4, & -3, & -4 \end{matrix}\right\},$$

which cancel each other, with the exception of

$$0,\ 0,\ 2,\ 2,\ 2,\ 2,\ -4,\ -4,\ -6,\ -6,\ -6,\ -6.$$

Theorems derived from the other Zeta Functions. § 68.

68. In this paper I have limited myself to results which may be regarded as generalizations of the formula

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) + 7\sigma(n-6) - 9\sigma(n-10) + \&c. = 0,$$

and of the corresponding formula (§ 44) which connects together the σ's of the first n numbers.

In a second paper, I propose to give the similar theorems which stand in a corresponding relation to the recurring formulæ

$$\zeta(n) + \zeta(n-1) + \zeta(n-3) + \zeta(n-6) + \zeta(n-10) + \&c. = 0,\text{*}$$

$$\Delta'(n) - 2\Delta'(n-1) + 2\Delta'(n-4) - 2\Delta'(n-9) + 2\Delta'(n-16) - \&c. = 0,\dagger$$

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[*] *Proc. Lond. Math. Soc.*, Vol. xv., p. 110, or *Proc. Camb. Phil. Soc.*, Vol. v., p. 116. The value of $\zeta(0)$ is supposed to be $-n$.

On pp. 118 and 119 of the former paper, and pp. 117 and 119 of the latter, the ζ-theorem corresponding to the σ-theorem in § 44 is given.

[†] The value of $\Delta'(0)$ is supposed to be $\frac{1}{2}n$. The corresponding theorem involving all the numbers from 1 to n as arguments is

$$\Delta'(n) - \Delta'(n-1) - \Delta'(n-2) - \Delta'(n-3) + \Delta'(n-4) + \Delta'(n-5) + \Delta'(n-6) + \Delta'(n-7)$$
$$+ \Delta'(n-8) - \Delta'(n-9) - \dots - \Delta'(n-15) + \Delta'(n-16) + \dots \pm \Delta'(0) = 0,$$

where $\Delta'(0)$ is to have the value $\frac{1}{2}r(r+1)$, $2r+1$ being the number of terms in the completed group in which $\Delta'(0)$ falls.

in which $\zeta(n)$ denotes the excess of the sum of the uneven divisors of n above the sum of the even divisors; and $\Delta'(n)$ denotes the sum of those divisors of n whose conjugates are uneven.

The general theorems relating to the actual divisors are derived from the other Zeta functions by a method similar to that employed in §§ 25–27.

On a certain Riemann's Surface. *By* W. BURNSIDE.

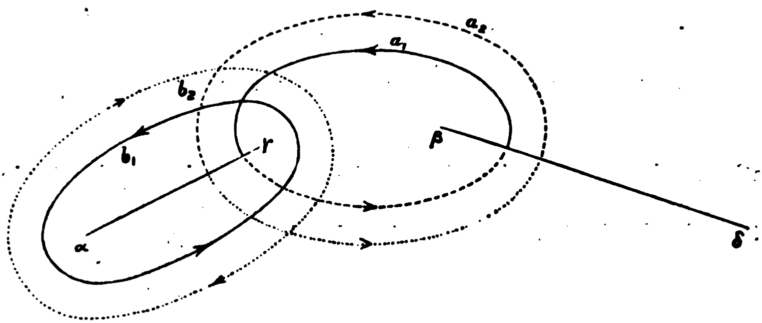
[*Read May* 14*th*, 1891.]

The present note was suggested by the paper of Mr. R. A. Roberts, pp. 28–34 of the current volume of the Society's *Proceedings*, dealing with a particular case of Abel's theorem. The special interest of the case that Mr. Roberts treats lies in the degradation of the apparently Abelian integrals to ordinary elliptic integrals; and though, as Mr. Greenhill has pointed out to me, this property, for the form of integral in question, is sufficiently well known, being originally due to Legendre, still its consideration from quite another point of view, namely, that of Riemann's theory, is perhaps not altogether superfluous.

The equation
$$s^3 = \frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}$$
defines a three-sheeted Riemann's surface. The only branch points are α, β, γ, δ, and at each of these the three sheets are connected together. The points at infinity on each sheet are ordinary points. The "Verzweigungs-schnitte," or lines of passage, will consist of a line from α to γ, connecting sheets I. and II., a second from α to γ, connecting II. and III., and two similar ones from β to δ. The lines may be taken straight, so that on crossing $\alpha\gamma$ or $\beta\delta$ from the left to the right a point will pass from I. to II., or from II. to III., or from III. to I.; the separate sheets being denoted by I., II., and III. The number of sheets being three, while there are four lines of passage, Riemann's number "p" for the surface will be 2, and there are, therefore, according to Riemann's theory, two independent integrals of the first kind upon it.

The normal series of sections, by which the surface is reduced to a simply connected one, will consist of two a-curves, two b-curves, and a line joining any point of the system a_1b_1 to any point of a_2b_2. A figure will indicate the system of curves chosen more readily than a verbal description, and in the figure the line joining a_1b_1 to a_2b_2, being of no importance for the integrals, is omitted.

Lines drawn in the first sheet are full, in the second broken, and in the third dotted.



The arrowheads on the a and b curves indicate the directions round them in which the integrals giving the periods will be taken.

The two simplest integrals of the first species on the surface are

$$\int \frac{dz}{s\,(z-\gamma)(z-\delta)} \text{ and } \int \frac{dz}{s^2\,(z-\gamma)(z-\delta)},$$

or

$$\int \frac{dz}{\{(z-\alpha)(z-\beta)(z-\gamma)^2(z-\delta)^2\}^{\frac{1}{3}}},$$

and

$$\int \frac{dz}{\{(z-\alpha)^2\,(z-\beta)^2\,(z-\gamma)(z-\delta)\}^{\frac{1}{3}}},$$

and they will be denoted by

$$\int dw \text{ and } \int dw'.$$

The periods of w and w', that is, the values of

$$\int dw \text{ and } \int dw',$$

taken round the sections, may be expressed in terms of rectilinear integrals in the following forms:—

$$\int_{b_1} dw = (1-T)\int_{\alpha}^{\gamma} dw, \qquad \int_{a_1} dw = (1-T)\int_{\beta}^{\gamma} dw,$$

$$\int_{b_2} dw = -T^2(1-T)\int_{\alpha}^{\gamma} dw, \qquad \int_{a_2} dw = T(1-T)\int_{\beta}^{\gamma} dw,$$

$$\int_{b_1} dw' = (1-T^2)\int_{\alpha}^{\gamma} dw', \qquad \int_{a_1} dw' = (1-T^2)\int_{\beta}^{\gamma} dw',$$

$$\int_{b_2} dw' = -T(1-T^2)\int_{\alpha}^{\gamma} dw', \qquad \int_{a_2} dw' = T^2(1-T^2)\int_{\beta}^{\gamma} dw',$$

where T is one of the imaginary cube roots of unity, $\int_{\alpha}^{\gamma}$ denotes integration in sheet I. along the right-hand side of $\alpha\gamma$, and $\int_{\beta}^{\gamma}$ denotes integration in sheet I.

Hence the constant differences of w and w', when the variable is restricted to the simply connected surface given by the sections, at corresponding points on opposite sides of the sections, are given by the table:—

	a_1	b_1	a_2	b_2
w	A	B	$-T^2A$	TB
w'	A'	B'	$-TA'$	T^2B'

or, in other words, the table gives the relations between the periods of the simple integrals.

The fundamental relation, connecting the periods of any two independent integrals of the first kind, gives, when applied to the above table,

$$AB'-A'B-T^2AB'+T^2A'B = 0,$$

or

$$TAB'+A'B = 0.$$

The normal integrals [illegible] kind, namely those whose constant differences at the a-curves are $(\pi i, 0)$ and $(0, \pi i)$ respectively, are given by

$$W_0 = \frac{\pi}{\sqrt{3}}\left(\frac{Tw}{A} - \frac{T^2 w'}{A'}\right),$$

and

$$W' = \frac{\pi}{\sqrt{3}}\left(\frac{w}{A} - \frac{w'}{A'}\right);$$

and, on substituting, it will be found that the periods of these normal integrals are given by the following table :—

	a_1	b_1	a_2	b_2
W	πi	$\frac{2\pi}{\sqrt{3}}\frac{B}{A}T$	0	$-\frac{\pi}{\sqrt{3}}\frac{B}{A}T$
W'	0	$-\frac{\pi}{\sqrt{3}}\frac{B}{A}T$	πi	$\frac{2\pi}{\sqrt{3}}\frac{B}{A}T$

the simplification arising from the previously proved relation

$$TAB' + A'B = 0.$$

An inspection of this table shows at once that W and W' are elliptic integrals, for the most general value of W is

$$W + m\pi i + (2n_1 - n_2)\frac{\pi}{\sqrt{3}}\,\frac{B}{A}\,.\,T,$$

and $2n_1 - n_2$ being an integer, the integral has therefore only two periods.

The ratio A/A', involved in the passage from w and w' to W and W', may be shown to be algebraical. For this purpose let b_1' be a closed curve on I, enclosing $\beta\delta$ and not cutting b_1; then, in the space outside b_1 and b_1', the integrand of w is everywhere finite, and hence

$$\int_{b_1} dw + \int_{b'_1} dw = 0.$$

Now the linear substitution, given by

$$\frac{z-a\,.\,z-\delta}{z-\beta\,.\,z-\gamma} = \frac{z'-a\,.\,z'-\delta}{z'-\beta\,.\,z'-\gamma},$$

simultaneously changes α, β, γ, δ into δ, γ, β, α, since it must be of the form

$$\frac{z-z_1}{z-z_2} = -\frac{z'-z_1}{z'-z_2},$$

for it is its own reciprocal.

Hence
$$z-\alpha = \frac{\{\tfrac{1}{2}(z_1+z_2)-\alpha\}}{D}(z'-\delta),$$

$$z-\beta = \frac{\{\tfrac{1}{2}(z_1+z_2)-\beta\}}{D}(z'-\gamma),$$

$$z-\gamma = \frac{\{\tfrac{1}{2}(z_1+z_2)-\gamma\}}{D}(z'-\beta),$$

$$z-\delta = \frac{\{\tfrac{1}{2}(z_1+z_2)-\delta\}}{D}(z'-\alpha),$$

where
$$D = z'-\tfrac{1}{2}(z_1+z_2);$$

therefore

$$dw = \frac{-\left(\frac{z_1-z_2}{2}\right)^2}{\left[\left(\frac{z_1+z_2}{2}-\alpha\right)\left(\frac{z_1+z_2}{2}-\beta\right)\left(\frac{z_1+z_2}{2}-\gamma\right)^2\left(\frac{z_1+z_2}{2}-\delta\right)^2\right]^{\frac{1}{3}}}$$

$$\times\frac{dz'}{\left[(z'-\alpha)^2(z'-\beta)^2(z'-\gamma)(z'-\delta)\right]^{\frac{1}{3}}}.$$

Also
$$\frac{z_1+z_2}{2} = \frac{\alpha\delta-\beta\gamma}{\alpha+\delta-\beta-\gamma},$$

$$\frac{z_1-z_2}{2} = \frac{\sqrt{(\alpha-\beta)(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)}}{\alpha+\delta-\beta-\gamma};$$

therefore
$$dw = \left(\frac{\alpha-\beta}{\delta-\gamma}\right)^{\frac{1}{3}} dw'.$$

and
$$\int dw = -\left(\frac{\alpha-\beta}{\delta-\gamma}\right)^{\frac{1}{3}}\int dw',$$

where the determination of the cube root will depend on which side of αγ the circular arc lies, into which βδ is changed by the substitution.

Now
$$\Delta' = (1-T^2)\int dw' = (T^2-1)\left(\frac{\delta-\gamma}{\alpha-\beta}\right)^{\frac{1}{3}}\int dw,$$

and
$$\Delta = (1-T)\int dw = (T-1)\int dw.$$

therefore $$A' = -\left(\frac{\delta-\gamma}{\alpha-\beta}\right)^{\frac{1}{3}} A,$$

with a proper determination of the root.

[*Addition. October* 30*th*, 1891.

The equation $$s^3 = \frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)} \text{.........................} (A),$$

defining the Riemann's surface considered in this note, can, by a rational and reversible transformation, be changed into the form

$$s^2 = z(1-z)(1-\kappa z)(1-\lambda z)(1-\kappa\lambda z) \text{ ....} (B),$$

where κ, λ are determinate in terms of α, β, γ, δ. But without performing this transformation the equations (A) and (B) have clearly this property in common: that each can be changed into itself by a transformation, of period two, which at the same time transforms into themselves two independent integrals of the first species.

Thus, for (A), if

$$\frac{(z-\alpha)(z-\gamma)}{(z-\beta)(z-\delta)} = \frac{(z'-\alpha)(z'-\gamma)}{(z'-\beta)(z'-\delta)}, \text{ and } s = \frac{1}{s'},$$

then $$s'^3 = \frac{(z'-\alpha)(z'-\beta)}{(z'-\gamma)(z'-\delta)},$$

and the two normal integrals considered in this paper are each unchanged; while for (B), if

$$z = \frac{1}{\kappa\lambda z'}, \text{ and } s = \frac{s'}{(\kappa\lambda)^{\frac{3}{2}} z'^3},$$

then $$s'^2 = z'(1-z')(1-\kappa z')(1-\lambda z')(1-\kappa\lambda z'),$$

and $\frac{1+\sqrt{\kappa\lambda}\,.\,z}{s}dz$, $\frac{1-\sqrt{\kappa\lambda}\,.\,z}{s}dz$ are unchanged except as regards sign.

If in either case the point (s, z) describes a period-path on the Riemann's surface, the point (s', z') must do the same, and a very little consideration shows that the corresponding paths are not reconcilable. Hence, at once, each integral can only have two independent periods. This reasoning may be applied to any equation

$$s^2 = f_6(z) \text{ ....} (C),$$

[where $f_6(s)$ is a sextic function,] which is such that it can be transformed into itself by a substitution, of period two, of the form

$$s = \frac{as'+b}{cs'+d}, \quad s = \Delta \frac{s'}{(cs'+d)^3},$$

and therefore the two integrals of the first species belonging to equation (C) can be expressed by means of elliptic integrals, if the equation

$$f_6(s) = 0$$

can be transformed into itself by such a linear substitution. The roots of the sextic are in this case in involution, and hence the condition is that the skew invariant should vanish. This is the result originally proved (in a different manner) by Prof. Gordan.]

Quartic Equations interpreted by the Parabola.

By G. Heppel, M.A.

[Read February 12th, 1891.]

The [illegible] solution of quartic equations by means of the parabola forms the subject of a paper by Hoppe in the Archiv der Mathematik, [illegible] and also of a paper by [illegible] in the [illegible] The object of the present paper is to [illegible] the method proposed in [illegible] supplementary to Hoppe's investigation.

[illegible] we derive from the quartic a [illegible] and if these [illegible] the quartic are at [illegible]

[illegible] may be effected by [illegible] variable [illegible] distance of the [illegible]

[illegible]

by removing the second term, and multiplying by y, we obtain a quartic, one root of which is zero, and the other three are the roots of the transformed cubic. Let this quartic be

$$y^4+qy^2+ry=0\,;$$

and let its roots represent the ordinates of the points of intersection of

$$(x-h)^2+(y-k)^2-c^2=0,$$

and

$$y^2=lx.$$

Then

$$y^4+qy^2+ry=0$$

is identical with

$$y^4+(l^2-2lh)\,y^2-2l^2ky+l^2\,(h^2+k^2-c^2)=0\,;$$

whence

$$h=(l^2-q)/2l,\quad k=-r/2l^2,$$

$$c^2=h^2+k^2,$$

the last of these equations not being at present made use of.

If a circle be drawn, having as centre $[h, k]$, and passing through the vertex of the parabola, it is necessary merely to measure the lengths of the ordinates to obtain at least one positive root of the original cubic before transformation.

Hence, according to the above plan, a quartic with the second term removed,

$$y^4+qy^2+ry+s=0,$$

is first treated algebraically by Descartes' solution, and the derived cubic solved geometrically by Hoppe's method.

We might also consider the four roots of this quartic as being identical with the values of the ordinates of the intersections with a fixed parabola of two straight lines making equal angles with the axis of x. It is, however, evident that four points may be joined, two and two, in three ways, and therefore the determination of any one of the three pairs of lines involves the solution of a cubic equation. In the paper by Hofmann this cubic is formed, and is supposed to be solved algebraically. A pair of straight lines is then drawn, and the roots of the quartic obtained by measurement.

It is proposed in the present paper to attempt to show how a purely geometrical method may be employed which renders it unnecessary to solve a cubic as Hofmann does, or to use Descartes' algebraical treatment as Hoppe does, and that by this method, Descartes' derived cubic may be arrived at by a geometrical road, its interpretation exhibited, and the four roots, whether real or imaginary, shown on one diagram, from which they may be measured. At the same time the whole process of Hoppe's solution of a cubic will be required as one step in the treatment of the quartic.

In order to do this, an extension of the primary machinery is necessary. Hitherto the parabola has been considered to have a latus rectum of arbitrary length, that is to say that the ratio between its length and the unit of measurement of the coefficients of the quartic has been arbitrary. Let this ratio be no longer considered arbitrary, but one to be determined in terms of the coefficients of the quartic. Then, making

$$y^4+qy^2+ry+s=0$$

identical with

$$y^4+(l^2-2lh)\,y^2-2l^2ky+l^2\,(h^2+k^2-c^2)=0,$$

we get, by eliminating h, k,

$$c^2=\frac{(l^2-q)^2}{4l^2}+\frac{r^2}{4l^4}-\frac{s}{l^2};$$

so that c^2 may have a variety of values according to the value assigned to l. Suppose

$$c=0,$$

the circle becomes a point with an imaginary portion consisting of two straight lines inclined at 45° to the axis of x, and in a plane perpendicular to that of the paper. Also since the equation to determine l^2 is

$$l^6-2ql^4+(q^2-4s)\,l^2+r^2=0,$$

this gives at least one negative value of l^2, thus determining an imaginary parabola, which may be interpreted as an equal parabola in a plane perpendicular to that of the paper.

Passing to this new plane, and considering it as a plane of reference, with coordinates x and y, it follows that two straight lines inclined at 45° to the axis of x may be found, provided a suitable value is given to l, such that the ordinates of the intersections of these lines with the parabola are the roots of

$$y^4+qy^2+ry+s=0.$$

The intersection of

$$(x-h)^2-(y-k)^2=0 \quad \text{with} \quad y^2=lx,$$

gives $$y^4-(2hl+l^2)\,y^2+2l^2ky+l^2\,(h^2-k^2)=0\,;$$

therefore $$q=-2hl-l^2, \quad r=2l^2k, \quad s=l^2\,(h^2-k^2),$$

and the elimination of h and k gives

$$l^6+2ql^4+(q^2-4s)\,l^2-r^2=0,$$

which is Descartes' cubic, and gives one or three real values of l, such that two straight lines at 45° to the axis of x may be found fulfilling the conditions we require.

The ordinates of the four points of intersection of the pair of lines and the parabola are

$$y=\frac{l}{2}\pm\sqrt{-\frac{l^2}{4}-\frac{q}{2}-\frac{r}{2l}},$$

and $$y=-\frac{l}{2}\pm\sqrt{-\frac{l^2}{4}-\frac{q}{2}+\frac{r}{2l}},$$

and four or two or none of these may be imaginary. The imaginary roots, if there are any, may be geometrically represented in the following manner. Let P be the point $[h,k]$. Let T be the point of intersection of tangents at the extremities of the latus rectum, or $\left[-\frac{l}{4},\,0\right]$. Join PT and produce it to Q, so that

$$TQ=PT.$$

Then, if through P, Q pairs of lines be drawn making angles of 45° with the axis of x, and if the line through P cuts the parabola in a point where

$$y=\pm\frac{l}{2}+z,$$

the parallel line through Q cuts it in a point where

$$y=\pm\frac{l}{2}+z\sqrt{-1}\,;$$

so that if the intersection of the first is real, that of the second is imaginary, and *vice versâ*.

Hence, considering space as divided into four compartments by the two tangents, if P is in the same compartment with the parabola, all the roots are real; if in the opposite one, all are imaginary; if in either of the others, two are real and two are imaginary. Two will be equal if P is upon a tangent.

Before proceeding to state in order the steps to be taken in treating a quartic equation in the manner here proposed, it may be as well to notice a slight modification of Hoppe's method, which may be found to simplify the preliminary transformation. The cubic to be solved being

$$y^3+\alpha y^2+\beta y+\gamma=0,$$

instead of removing the second term and multiplying by y, it may perhaps be simpler to multiply the original equation by $y-\alpha$, giving

$$y^4+(\beta-\alpha^2)\,y^2+(\gamma-\alpha\beta)\,y-\alpha\gamma=0.$$

The centre of the circle is then

$$h=(l^2+\alpha^2-\beta)/2l,\quad k=(\alpha\beta-\gamma)\,/\,2l^2,$$

and the circle must pass through a point in the parabola whose ordinate is α.

In the practical application of this method, it would of course be very troublesome to draw to scale a fresh parabola for every equation requiring solution. It is quite possible to determine all the roots to a fair degree of approximation, without actually drawing a line, by means of a single parabola drawn on a surface divided into small squares, where distances between points may be measured by strips of paper, and ordinates may be read off by means of the small squares. In the solution of the derived cubic, we are at liberty to choose any convenient unit of measurement, and thus the value of l, which will be required to be found before the diagram of the roots of the quartic can be drawn, can be easily determined. It is not quite the same thing when the fixed parabola is used for the diagram with the two straight lines. It might happen, for example, that the length h or the length k might come out a thousand times that of the latus rectum. Here it would be necessary to transform the quartic equation to one having roots less than those of the original equation, in some convenient ratio. Practically, however, a parabola with a latus rectum of two inches, drawn on a surface divided into small squares with sides of a tenth of an inch in length, will be found generally suitable.

Supposing such an arrangement, the following is an example of Hoppe's solution of a cubic.

Let the cubic be $\quad y^3-7y^2+80y-311=0.$

Here $$h = (l^2+\alpha^2-\beta)/2l = \tfrac{1}{2}l-31/2l,$$
$$k = (\alpha\beta-\gamma)/2l^2 = -249/2l^2.$$

Suppose the unit of measurement is four-tenths of an inch, so that
$$l = 5;$$
then $h = -{\cdot}6$ units or $-2{\cdot}4$ tenths, $k = -4{\cdot}98$ units or $-19{\cdot}92$ tenths. One point of intersection has an ordinate -7 units or -28 tenths. The measurement of the ordinate of the one other real point approximates to the true value, which is 18·08 tenths, and the root of the cubic is 4·52.

Example of the Diagram for a Quartic.

Let the quartic be $y^4-14y^2-23y+68 = 0.$

The cubic for l^2 is $l^6-28l^4-76l^2-529 = 0.$

This will be found by Hoppe's method to have a root
$$l^2 = 31;$$
therefore $$l = 5{\cdot}57.$$

The unit of measurement is therefore $\dfrac{20}{5{\cdot}57}$ tenths.

The point P is $$x = -\frac{5{\cdot}57}{2}+\frac{14}{11{\cdot}14} \text{ units,}$$
or $$-10+\frac{140}{31} \text{ or } -5{\cdot}48 \text{ tenths;}$$
$$y = -\frac{23}{62} \text{ units, or } -\frac{230}{172{\cdot}67}, \text{ or } -1{\cdot}33 \text{ tenths.}$$

The point being neither in the parabola compartment nor in the one opposite, shows that two roots are real and two are imaginary.

The point T being $[-5, 0]$; the point Q is $[-4{\cdot}52, 1{\cdot}33]$. Applying a straight edge at an angle of 45° through P gives the distance of the point of intersection from the diameter through the upper extremity of the latus rectum as 4·12 tenths, and applying it at an angle of 135° through Q gives the distance from the diameter through the lower extremity as 6·02.

$$4{\cdot}12\times5{\cdot}57\div20 = 1{\cdot}15, \quad 6{\cdot}02\times5{\cdot}57\div20 = 1{\cdot}68,$$

so that the roots are

$$2{\cdot}785 \pm 1{\cdot}15 \quad\text{and}\quad -2{\cdot}785 \pm 1{\cdot}68\sqrt{-1}.$$

In conclusion, it may be interesting to prove that the criteria of the nature of the roots of the quartic equation, which depend upon the compartment in which P is situated, correspond exactly to those obtained by algebraical analysis.

From the compartment test we see that two roots will be equal if

$$\left(\frac{l}{4}+\frac{q}{2l}\right)^2 = \frac{r^2}{4l^4};$$

and all real or all imaginary if

$$\left(\frac{l}{4}+\frac{q}{2l}\right)^2 > \frac{r^2}{4l^4}.$$

The nature of the roots therefore depends upon whether

$$l^8+4ql^6+4q^2l^4-4r^2$$

is equal to, greater than, or less than 0; or, since

$$l^6+2ql^4+(q^2-4s)\,l^2-r^2=0,$$

upon whether $16s-4ql^2-3l^4$ is zero, positive, or negative. Let

$$16s-4ql^2-3l^4 = M.$$

Then the elimination of l^2 from

$$M=0 \quad\text{and}\quad l^6+2ql^4+(q^2-4s)\,l^2-r^2=0,$$

gives $$256s^3-128q^2s^2+16q^4s+144qr^2s-4q^3r^2-27r^4=0,$$

or $$\Delta = 0,$$

where Δ is the discriminant of the quartic y^4+qy^2+ry+s.

Substitute, in Δ, $l^6+2ql^4+(q^2-4s)\,l^2$ for r^2,

so that Δ is expressed as a function of q, s, and l. Then, since M and Δ vanish together, M is a factor of Δ; and by division the other factor is found to be

$$\{4s-q^2-4ql^2-3l^4\}^2.$$

Hence M and Δ have the same sign. Therefore,

if Δ is positive the roots are all real or all imaginary;

if Δ is negative the roots are two real and two imaginary.

Supposing the first alternative to be true, the point P will be in the parabola compartment, or the opposite one, according as $\frac{l}{2}+\frac{q}{2l}$ is less or greater than $\frac{l}{4}$, or as l^2+2q is negative or positive. Hence, if q is positive, there is no doubt that all the roots must be imaginary. If q is negative further investigation is necessary.

Now
$$4s-q^2 = l^2(l^2+2q)-r^2/l^2;$$
therefore, if l^2+2q is negative, $4s-q^2$ is negative.

Again,
$$\frac{r^2}{4l^4} \text{ is less than } \left(\frac{l}{4}+\frac{q}{2l}\right)^2,$$
since M is by hypothesis positive; therefore
$$\frac{r^2}{l^2} \text{ is less than } \left(\frac{l^2}{2}+q\right)^2;$$
therefore
$$4s-q^2 \text{ is greater than } l^2(l^2+2q)-(\tfrac{1}{2}l^2+q)^2.$$
$$16s-4q^2 \text{ is greater than } (3l^2-2q)(l^2+2q);$$
so that if l^2+2q is positive, $4s-q^2$ is positive.

Consequently, if Δ is positive, the roots are all imaginary, unless q is negative, and at the same time $4s-q^2$ is negative. These results exactly agree with those given in the treatise on the "Theory of Equations," by Burnside and Panton.

June 11*th*, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

Prof. H. Lamb was admitted into the Society.

The following communications were made:—

Systems of Spherical Harmonics: Mr. E. W. Hobson.

On the Motion of a Liquid Ellipsoid under its own Attraction: Dr. M. J. M. Hill.

On certain Properties of Symmetric, Skew Symmetric, and Orthogonal Matrices: Dr. H. Taber (communicated by Mr. Hammond).

An Application of the Method of Images to the Conduction of Heat: Mr. G. H. Bryan.

A Property of the Circum-circle: Mr. R. Tucker.

Messrs. Walker, Lamb, and the President, took part in discussions on the papers.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xlix., No. 299.

"Proceedings of the Physical Society of London," Vol. xi., Part i., April, 1891.

"Educational Times," for June.

"Canadian Institute—Fourth Annual Report, Transactions," Vol. i., Part ii., March, 1891; Toronto.

"Time Reckoning for the Twentieth Century," by Sandford Fleming, 8vo pamphlet; Washington, 1889.

"Observations made during the year 1885 at the United States Naval Observatory," 4to; Washington, 1891.

"Bulletin des Sciences Mathématiques," April, 1891.

"Journal für die reine und angewandte Mathematik," Band cviii., Heft 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vii., Fasc. viii.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 129, 130, and two Nos. of Index.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang xxxv., 3$^{\text{es}}$ und 4$^{\text{es}}$ Hefte.

"Leçons sur la Théorie Générale des Surfaces," Part iii.; par Gaston Darboux.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stück 5.

An Application of the Method of Images to the Conduction of Heat.

By G. H. Bryan.

[*Read June 11th*, 1891.]

1. In his interesting and suggestive communication on "Synthetical Solutions in the Conduction of Heat" (*Proceedings*, Vol. xix., pp. 285–294), Mr. Hobson has investigated the [illegible] heat in an infinite solid bounded [illegible] across which [illegible] place, and initially heated in [illegible]. He [illegible] effect

of the bounding plane may be represented by a system of images, the initial distribution at any point being replaced by a line of sources extending from that point to its geometrical image together with a line of sinks extending from that image to infinity.

It can be shown that when the distributions corresponding to all the elements are taken collectively, Mr. Hobson's expression, on integration, leads to the given initial distribution of temperature, provided that the latter is everywhere finite and continuous. But a difficulty arises if the initial distribution of temperature contains any singular points, such as sources or doublets, or is discontinuous in any part of the body.

For example, the expression appears to fail when the temperature is due to a simple source of heat at any point. A solution applicable to such a case must be obtained by integrating Mr. Hobson's expression by parts. The problem, however, appeared to me to be of sufficient interest to merit an independent solution. Accordingly, in the present paper I have investigated expressions for the images of a single instantaneous source of heat generated at any point within the body, and have thus obtained a form of solution which holds for any given initial distribution whatever.

2. As the mode of solution is almost identical whether we consider the problem as one, two, or three-dimensional, I shall consider the general case, where the motion takes place in n dimensions. It will then only be necessary to put $n = 1, 2$, or 3, to obtain the solution for a bar, plate, or solid.

Suppose the body extending from $x = 0$ to $x = +\infty$, and let h be the coefficient of surface-radiation at the boundary $x = 0$. Let us suppose a quantity Q of heat instantaneously generated at the point $x = a$ on the axis of x at the time $t = 0$. If the body were unlimited in every direction the temperature at any other position and time would be given by

$$v_1 = \frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}} \exp - \frac{(x-a)^2+y^2+\&c.}{4\kappa t} \quad\dots\dots\dots\dots\dots\dots (1).$$

Owing, however, to the boundary, the temperature will be different. Let it be v_1+v_2, where v_1 is the expression just written down, and v_2 is the temperature due to a system of images which represent the effect of the plane.

The boundary condition, at $x = 0$, gives

$$\frac{d(v_1+v_2)}{dx} - h(v_1+v_2) = 0 \quad\dots\dots\dots\dots\dots\dots (2);$$

therefore $$\frac{dv_2}{dx} - hv_2 = -\left(\frac{dv_1}{dx} - hv_1\right)$$

$$= \frac{Q}{(4\pi\kappa t)^{\frac{3}{2}}}\left\{\frac{x-a}{2\kappa t} + h\right\}\exp - \frac{(x-a)^2 + \&c.}{4\kappa t},$$

when $x = 0$.

Whence it is evident that we may take

$$\frac{dv_2}{dx} - hv_2 = \frac{Q}{(4\pi\kappa t)^{\frac{3}{2}}}\left\{-\frac{x+a}{2\kappa t} + h\right\}\exp - \frac{(x+a)^2 + \&c.}{4\kappa t}$$

$$= \left(\frac{d}{dx} + h\right)\left\{\frac{Q}{(4\pi\kappa t)^{\frac{3}{2}}}\exp - \frac{(x+a)^2 + \&c.}{4\kappa t}\right\};$$

and the value of v_2 determined by this equation will satisfy the equation of conduction. The last equation may be written

$$\frac{d}{dx}(e^{-hx}v_2) = \frac{d}{dx}\left\{e^{-hx}\frac{Q}{(4\pi\kappa t)^{\frac{3}{2}}}\exp - \frac{(x+a)^2 + \&c.}{4\kappa t}\right\}$$

$$+ 2he^{-hx}\frac{Q}{(4\pi\kappa t)^{\frac{3}{2}}}\exp - \frac{(x+a)^2 + \&c.}{4\kappa t};$$

therefore, integrating, since v_2 is not infinite when $x = -\infty$, we have

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

the element $d\xi$ at the point $x = -(a+\xi)$ is

$$2hQe^{-h\xi}\,d\xi.$$

If the boundary $x=0$ is *adiathermanous*, then $h=0$, and the images reduce to a single source Q at $x = -a$, as they should do.

3. The complete value of the temperature due to a source Q at the point (a, b, &c.), is therefore

$$v = v_1 + v_2$$

$$= \frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}}\left\{\exp - \frac{(x-a)^2+(y-b)^2+\&c.}{4\kappa t} + \exp - \frac{(x+a)^2+(y-b)^2+\&c.}{4\kappa t}\right\}$$

$$- \int_0 \frac{2hQe^{-h\xi}}{(4\pi\kappa t)^{\frac{1}{2}n}} \exp - \frac{(x+a+\xi)^2+(y-b)^2+\&c.}{4\kappa t}\,d\xi \;\ldots\; (5).$$

From this expression, the temperature corresponding to any given initial distribution may be immediately written down. It will be only necessary here to mention the principal cases of interest.

(i.) If the initial temperature of the point (a, b, &c.) is $F(a, b, \&c.)$, we have only to write $F(a, b, \&c.)\,da, db \ldots$ for Q, and integrate over the volume of the body.

(ii.) For an instantaneous doublet at (a, b, &c.), we have to differentiate with respect to a if the doublet is parallel to the axis of x, and with respect to b if parallel to the axis of y.

(iii.) For a continuous source at (a, b, ...), existing from the time $t = 0$, we must write $Qd\lambda$ for Q, and $t-\lambda$ for t, and integrate from $\lambda = 0$ to $\lambda = t$. We may suppose Q either constant or a function of λ.

(iv.) Suppose the body is bounded by the *two* rectangular planes $x = 0$ and $y = 0$, across which it is radiating into temperature zero, and let the coefficients of surface radiation at $x = 0$ and $y = 0$ be h and h' respectively. Then it is easy to see that a source Q of heat instantaneously generated at the point (a, b), gives rise to the same distribution of temperature as would be produced in a body infinite in all directions by the following sources and images :—

(1) Four equal sources $+Q$ at the points ($\pm a$, $\pm b$).

(2) Two lines of sinks extending from $x = -a$ to $x = -\infty$ along the lines $y = \pm b$, such that the intensity of the sink in the element dx of either line is

$$2hQe^{h(x+a)}\,dx.$$

(3) Two lines of sinks extending from $y=-b$ to $y=-\infty$ along the lines $x=\pm a$, such that the intensity of the sink in the element dy of either line is

$$2h'Qe^{h'(y+b)}\,dy.$$

(4) A sheet of sources extending from $x=-a$ to $x=-\infty$, and from $y=-b$ to $y=-\infty$, such that the intensity of the source in the element $dx\,dy$ is

$$4hh'Qe^{h(x+a)+h'(y+b)}\,dx\,dy.$$

(v.) The three-dimensional problem, in which the body is bounded by the planes $x=0$, $y=0$, $z=0$, may be treated in the same way. The effect of a single source at the point (a, b, c) will be represented by a distribution consisting of

(1) Eight equal sources, one in each quadrant.

(2) Twelve lines of sinks, four parallel to each axis.

(3) Six sheets of sources, two parallel to each coordinate plane.

(4) A solid of sinks—*i.e.*, an initial distribution of negative temperatures extending over the octant bounded by the planes $x=-a$, $y=-b$ and $z=-c$.

4. By Taylor's theorem the expression for v in (5) may be thrown into the form

$$v=\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}}\left\{\exp-\frac{(x-a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}+\exp-\frac{(x+a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}\right\}$$

$$-\int_0^\infty 2he^{-h\xi}\sum_{r=0}^{r=\infty}\frac{\xi^r}{r!}\left(\frac{d}{da}\right)^r\left\{\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}}\exp-\frac{(x+a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}\right\}d\xi.$$

Remembering that $\int_0^\infty \xi^r e^{-h\xi}\,d\xi=\frac{r!}{h^{r+1}}$, this gives

$$v=\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}}\left\{\exp-\frac{(x-a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}-\exp-\frac{(x+a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}\right\}$$

$$-2\sum_{r=1}^{r=\infty}\frac{1}{h^r}\left(\frac{d}{da}\right)^r\left\{\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}}\exp-\frac{(x+a)^2+(y-b)^2\text{ \&c.}}{4\kappa t}\right\}\;\ldots\ldots\;(6).$$

In this expression the line of images is replaced by an infinite series of multiple sources or singular points occurring at the point $(-a, b, \&c.)$, *i.e., at the geometrical image of the source of heat. The intensity of the multiple source of order* r *is* $-2Q/h^r$.

This mode of representation is very convenient when the radiation takes place very freely, so that h is very great; in such cases, a few terms of the series will be sufficient for approximate calculations. If $h = \infty$, the series vanishes, and the images reduce to a source $-Q$ at the geometrical image of the source Q, as is otherwise evident, for the boundary $x = 0$ is then at temperature zero.

5. There exists an infinite number of initial distributions in which the number of images becomes finite. For operating on v with $1-\frac{1}{h}\frac{d}{da}$ in (6), we obtain the solution

$$v' = \left(1-\frac{1}{h}\frac{d}{da}\right)\left\{\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}} \exp - \frac{(x-a)^2+(y-b)^2 \text{ \&c.}}{4\kappa t}\right\}$$

$$-\left(1+\frac{1}{h}\frac{d}{da}\right)\left\{\frac{Q}{(4\pi\kappa t)^{\frac{1}{2}n}} \exp - \frac{(x+a)^2+(y-b)^2 \text{ \&c.}}{4\kappa t}\right\} \quad \text{......} (7).$$

Hence an initial distribution, consisting of a source Q and a doublet $-Q/h$ at the point $(a, b, \text{\&c.})$, will have for its image a source $-Q$, and a doublet $+Q/h$ at the point $(-a, b, \text{\&c.})$.

It is evident in like manner that a source Q and a multiple point, of order m and intensity $-Q/h^m$, will have for their images a finite number of multiple sources, of order not exceeding m. More generally two multiple sources of orders m, m', and intensities inversely proportional to h^m and $-h^{m'}$, will give rise to an image consisting of $m'-m+1$ multiple points, of orders ranging from m to m'.

6. Another interesting distribution is that due to a source and a sink situated on the axis of x at the points $x = a$ and $x = a'$, whose intensities are in the ratio $e^{-ha} : e^{-ha'}$, *i.e.*, $e^{h(a-a')} : 1$. Let us call the intensities qe^{-ha} and $qe^{-ha'}$ respectively. Then it is evident from § 2, that the images will consist of a source qe^{-ha} at $(-a, 0)$, a sink $qe^{-ha'}$ at $(-a', 0)$, and a *finite* line of sinks, of intensity $2hqe^{hx}$, extending from $x = -a$ to $x = -a'$. Thus the images will be limited to the portion of the axis of x intercepted between the geometrical images of the given source and sink.

Numerous other cases might be found in which the distribution of images is finite or limited, but it would hardly be interesting to pursue the subject further in this direction.

[*Addition, November* 11*th*, 1891.

7. As an example, let us apply the image method to determine the temperature at any time at the surface of an infinite solid bounded by a radiating plane, and initially heated throughout to a uniform temperature v_0.

This is the problem solved by Riemann in his *Partielle Differentialgleichungen*, § 69, by considering the solid as the limit of a sphere whose radius is infinitely large.

The flow of heat is evidently one-dimensional. Writing $v_0\,da$ for Q in the formula (6) of § 4, and integrating between the limits 0 and ∞, we have for the temperature the expression

$$v = \frac{v_0}{2\sqrt{(\pi\kappa t)}}\int_0^\infty \left\{\exp -\frac{(x-a)^2}{4\kappa t} - \exp -\frac{(x+a)^2}{4\kappa t}\right\} da$$

$$-\frac{2v_0}{2\sqrt{(\pi\kappa t)}}\left[\sum_{r=1}^{r=\infty}\frac{1}{h^r}\left(\frac{d}{da}\right)^{r-1}\exp -\frac{(x+a)^2}{4\kappa t}\right]_{a=0}^{a=\infty}$$

$$= \frac{v_0}{\sqrt{\pi}}\int_{-x/2\sqrt{(\kappa t)}}^{x/2\sqrt{(\kappa t)}} e^{-u^2}du + \frac{v_0}{\sqrt{(\pi\kappa t)}}\sum_{r=1}^{r=\infty}\frac{1}{h^r}\left(\frac{d}{dx}\right)^{r-1}\exp -\frac{x^2}{4\kappa t}.$$

To find the temperature at the surface, we must put $x = 0$ in the above expression. Hence the term involving the integral vanishes. Also, if

$$f(x) = \exp -\frac{x^2}{4\kappa t},$$

we have, by Maclaurin's theorem,

$$\sum_{r=0}^{r=\infty}\frac{x^r}{r!}f^{(r)}(0) = f(x) = \exp -\frac{x^2}{4\kappa t} = \sum_{m=0}^{m=\infty}\frac{(-1)^m}{m!}\,\frac{x^{2m}}{(4\kappa t)^m};$$

whence if r be odd, $\qquad f^{(r)}(0) = 0,$

and if r be even, and $= 2m$,

$$f^{(2m)}(0) = \frac{(-1)^m}{(4\kappa t)^m}\,\frac{(2m)!}{m!}.$$

Therefore, on substitution, we have

$$v = \frac{v_0}{h\sqrt{(\pi\kappa t)}}\sum_{m=0}^{m=\infty}\frac{(-1)^m\,(2m)!}{h^{2m}\,(4\kappa t)^m\,m!}$$

$$= \frac{v_0}{h\sqrt{(\pi\kappa t)}}\left\{1 - \frac{1}{2h^2\kappa t} + \frac{1.3}{(2h^2\kappa t)^2} - \frac{1.3.5}{(2h^2\kappa t)^3} + \ldots\right\}.$$

When t is very great this gives approximately

$$v = \frac{v_0}{h\sqrt{(\pi\kappa t)}}.$$

This result agrees with that found by Riemann.]

Systems of Spherical Harmonics. *By* E. W. HOBSON.

[*Read June* 11*th*, 1891.]

The following communication has been suggested by a study of the theories of spherical harmonics given by Thomson and Tait, and by Maxwell. I have, first of all, considered the most general solution of Laplace's equation, which is of degree zero, and have then shown how, from this, or the corresponding harmonics of degree -1, harmonics of any integral degree may be derived by differentiation with respect to a number of axes. I have further shown that all harmonics may be derived from harmonics of degree -1 by successive differentiation with respect to one axis only; new expressions for the ordinary tesseral harmonics have been obtained from this theorem.

A theory of poles for the zonal and tesseral harmonics of the second kind is given, analogous to Maxwell's theory for those of the first kind. I have considered classes of harmonics which involve the circular measure of the azimuthal angle ϕ, and which I have ventured to call "circulatory harmonics." In the last section, I have obtained definite integral expressions for the most general harmonics of given order, irrespective of degree, and have also given series involving an arbitrary function, which express the most general harmonics of the first kind, of given order.

The most general harmonics of given positive or negative degree.

1. If in Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which becomes, when the polar coordinates r, θ, ϕ are taken as independent variables,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0,$$

we put

$$d\chi=\operatorname{cosec}\theta\, d\theta,$$

or

$$\chi=\log\tan\tfrac{1}{2}\theta=\log\sqrt{\frac{r-z}{r+z}},$$

the equation becomes

$$\sin^2\theta\,\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial\chi^2}+\frac{\partial^2 V}{\partial\phi^2}=0\,;$$

if now $V = V_0$,

a function of ϕ and χ only, V_0 satisfies the equation

$$\frac{\partial^2 V_0}{\partial\chi^2}+\frac{\partial^2 V_0}{\partial\phi^2}=0.$$

The solution of this equation is

$$V_0=f(\chi+\iota\phi)+F(\chi-\iota\phi)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

which may also be written

$$V_0=F(e^{\iota\phi}\tan\tfrac{1}{2}\theta)+\psi(e^{-\iota\phi}\tan\tfrac{1}{2}\theta)\ldots\ldots\ldots\ldots\ldots\ldots(2)\,;$$

either of these expressions gives the most general spherical harmonic of degree zero. This solution was, I find, first given by Donkin;* the result may be expressed by saying that all spherical harmonics of degree zero are obtained by taking conjugate functions of the two quantities

$$\tan^{-1}\frac{y}{x}\ \text{and}\ \log\sqrt{\frac{r-z}{r+z}}.$$

From the form (2), Donkin proceeds to find the expression for the most general surface harmonic of degree n or $-n-1$, in the form

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left[F(e^{\iota\phi}\tan\tfrac{1}{2}\theta)+\psi(e^{-\iota\phi}\tan\tfrac{1}{2}\theta)\right];$$

this form is, however, not adapted for the discussion of various types of harmonics, and it is better, for this purpose, to use the method adopted by Thomson and Tait, and by Maxwell, which consists in differentiating harmonics of degree -1 with respect to the quantities x, y, z.

If we replace, in Donkin's expression for V_0, the quantities θ, ϕ by their values in terms of x, y, z, we obtain the form

$$V_0=F\left(\frac{x+\iota y}{r+z}\right)+\psi\left(\frac{x-\iota y}{r+z}\right)$$

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* See *Phil. Trans.*, 1857.

thus values of V_0 are obtained by taking conjugate functions of the two quantities

$$\frac{x}{r+z}, \quad \frac{y}{r+z}.$$

The most general harmonic of degree -1 is

$$\frac{V_0}{r} \text{ or } \frac{1}{r} f\left(\frac{x \pm \iota y}{r+z}\right);$$

differentiating this last expression with respect to n axes $h_1, h_2 \dots h_n$, we find the expression

$$\frac{d^n}{dh_1,\, dh_2 \dots dh_n} \left[\frac{1}{r} f\left(\frac{x \pm \iota y}{r+z}\right)\right] \dots\dots\dots\dots\dots\dots\dots (3),$$

where $\frac{d}{dh_r}$ denotes $l_r \frac{d}{dx} + m_r \frac{d}{dy} + n_r \frac{d}{dz}$, as a harmonic of degree $-(n+1)$; the corresponding harmonic of degree n is

$$r^{2n+1} \frac{d^n}{dh_1,\, dh_2 \dots dh_n} \left[\frac{1}{r} f\left(\frac{x \pm \iota y}{r+z}\right)\right].$$

Instead of differentiating harmonics of degree -1, we might have differentiated harmonics of degree zero, and thus have obtained the expression

$$r^{2n+1} \frac{d^{n+1}}{dh_1,\, dh_2 \dots dh_{n+1}} \left[f\left(\frac{x \pm \iota y}{r+z}\right)\right].$$

These two expressions for harmonics of degree n are really the same.

I shall now show that if, in the expression (3), all the axes be taken to coincide with the axis of z, there is no loss of generality, and thus the most general harmonic of degree $-n-1$ is

$$\frac{\partial^n}{\partial z^n} \left[\frac{1}{r} f\left(\frac{x \pm \iota y}{r+z}\right)\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots (4).$$

If V_n is a harmonic of positive or negative degree n, it is possible to choose u_{n+1} as a function of x and y of the $n+1$th degree, so that

$$\int_0^z V_n dz + u_{n+1}$$

is a spherical harmonic. We have

$$\nabla^2\left\{\int_0^z V_n dz+u_{n+1}\right\}=\int_0^z\left(\frac{\partial^2 V_n}{\partial x^2}+\frac{\partial^2 V_n}{\partial y^2}\right)dz+\frac{\partial V_n}{\partial z}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_{n+1}$$

$$=\frac{\partial V_n}{\partial z}-\int_0^z\frac{\partial^2 V_n}{\partial z^2}dz+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_{n+1}$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_{n+1}+\left(\frac{\partial V_n}{\partial z}\right)_{z=0}$$

If $\xi=x+\iota y$, $\eta=x-\iota y$, we have to choose u_{n+1}, so that

$$4\frac{\partial^2 u_{n+1}}{\partial\xi\partial\eta}=-\left(\frac{\partial V_n}{\partial z}\right)_{z=0}$$

The right-hand side of this equation is a function of ξ, η of degree $n-1$, hence the value required of u_{n+1} is

$$\iint-\tfrac{1}{4}\left(\frac{\partial V_n}{\partial z}\right)_{z=0}d\xi d\eta.$$

If any value of this expression be taken for u_{n+1}, we have

$$\int_0^z V_n\,dz+u_{n+1},$$

a harmonic V_{n+1} of degree $n+1$, such that

$$V_n=\frac{\partial V_{n+1}}{\partial z}.$$

It follows that, corresponding to any harmonic $V_{-(n+1)}$ of degree $-(n+1)$, there are three V^1_{-n}, V^2_{-n}, V^3_{-n} of degree $-n$, such that

$$V_{-(n+1)}=\frac{\partial}{\partial x}V^1_n=\frac{\partial}{\partial y}V^2_n=\frac{\partial}{\partial z}V^3_n,$$

$$\left[\text{Ex.: }\frac{x}{r^3}=-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\partial}{\partial y}\left\{\frac{xy}{(x^2+z^2)\,r}\right\}=\frac{\partial}{\partial z}\left\{\frac{xz}{(x^2+y^2)\,r}\right\}\right].$$

By integrating any harmonic of degree $-(n+1)$, n times with respect to z, we shall obtain a harmonic of degree -1; therefore *the most general harmonic of degree* $-(n+1)$ *is*

$$\frac{\partial^n}{\partial z^n}\left[\frac{1}{r}f\left(\frac{x\pm\iota y}{r+z}\right)\right]\quad\text{or}\quad\frac{\partial^{n+1}}{\partial z^{n+1}}\left[f\;\text{[illegible]}\;y)\right]\text{............(4).}$$

[We see at once that

$$\frac{\partial}{\partial z} f\left(\frac{x \pm \iota y}{r+z}\right) = -\frac{1}{r} f'\left(\frac{x \pm \iota y}{r+z}\right);$$

thus the two expressions are really the same].

It has thus been shown that any harmonic of degree $-(n+1)$ may be obtained by differentiating some harmonic of degree -1, n times with respect to z only.

Table of Harmonics of degree zero.

2. In the following table, a number of the most important harmonics of degree zero is given, with the conjugate functions of ϕ and χ from which they are derived.

HARMONICS.	VALUE OF $f(\chi \pm \iota\phi)$.
1	1
$\log_e \sqrt{\frac{r-z}{r+z}}, \quad \tan^{-1}\frac{y}{x}$	$\chi \pm \iota\phi$
$\left(\tan^{-1}\frac{y}{x}\right)^2 - \left(\log\sqrt{\frac{r-z}{r+z}}\right)^2, \quad \tan^{-1}\frac{y}{x}\log\sqrt{\frac{r-z}{r+z}}$	$(\chi \pm \iota\phi)^2$
$\frac{x(r-z)}{x^2+y^2} = \frac{x}{r+z}, \quad \frac{y(r-z)}{x^2+y^2} = \frac{y}{r+z}$	$e^{\chi \pm \iota\phi}$
$\frac{x(r+z)}{x^2+y^2} = \frac{x}{r-z}, \quad \frac{y(r+z)}{x^2+y^2} = \frac{y}{r-z}$	$e^{-(\chi \pm \iota\phi)}$
$\frac{(r+z)^m}{(x^2+y^2)^{\frac{1}{2}m}} \begin{matrix}\cos\\ \sin\end{matrix} m\phi, \quad \frac{(r-z)^m}{(x^2+y^2)^{\frac{1}{2}m}} \begin{matrix}\cos\\ \sin\end{matrix} m\phi$	$e^{\pm m(\chi \pm \iota\phi)}$
$\left.\begin{matrix} \frac{2zy}{x^2+y^2}\tan^{-1}\frac{y}{x} - \frac{xr}{x^2+y^2}\log\frac{r+z}{r-z} \\ \frac{2zx}{x^2+y^2}\tan^{-1}\frac{y}{x} + \frac{yr}{x^2+y^2}\log\frac{r+z}{r-z} \\ \frac{2ry}{x^2+y^2}\tan^{-1}\frac{y}{x} - \frac{xz}{x^2+y^2}\log\frac{r+z}{r-z} \\ \frac{2rx}{x^2+y^2}\tan^{-1}\frac{y}{x} + \frac{yz}{x^2+y^2}\log\frac{r+z}{r-z} \end{matrix}\right\}$	$(\chi \pm \iota\phi)\, e^{(\chi \pm \iota\phi)}$ $(\chi \pm \iota\phi)\, e^{-(\chi \pm \iota\phi)}$

HARMONICS.	VALUE OF $f(\chi \pm \iota\phi)$.
$\frac{(r-z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\left[2\tan^{-1}\frac{y}{x}\cos m\phi - \sin m\phi \log\frac{r+z}{r-z}\right]$ $\frac{(r-z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\left[2\tan^{-1}\frac{y}{x}\sin m\phi + \cos m\phi \log\frac{r+z}{r-z}\right]$	$(\chi \pm \iota\phi)\, e^{m(\chi \pm \iota\phi)}$
$\frac{(r+z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\left[2\tan^{-1}\frac{y}{x}\cos m\phi + \sin m\phi \log\frac{r+z}{r-z}\right]$ $\frac{(r+z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\left[2\tan^{-1}\frac{y}{x}\sin m\phi - \cos m\phi \log\frac{r+z}{r-z}\right]$	$(\chi \pm \iota\phi)\, e^{-m(\chi \pm \iota\phi)}$

The corresponding harmonics of degree -1 are obtained by multiplying the above by $\frac{1}{r}$.

Besides the ordinary harmonics obtained by differentiating $\frac{1}{r}$, after Maxwell's fashion, with regard to a number of axes, I shall consider the systems obtained by differentiating the harmonics

$$\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}},\quad \frac{1}{r}\tan^{-1}\frac{y}{x},\quad \frac{1}{r}\tan^{-1}\frac{y}{x}\log\sqrt{\frac{r+z}{r-z}}.$$

The System of Point-Harmonics.

3. If $\frac{1}{r}$ is differentiated with respect to n axes coinciding with the axis of z, the zonal harmonic

$$\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{2^n n!}\frac{\partial^n}{\partial z^n}\frac{1}{r} \quad \dots\dots\dots\dots\dots\dots\dots (5),$$

is obtained. If only $n-m$ axes coincide with the axis of z, and the remaining m axes are distributed symmetrically in the equatorial plane, the corresponding harmonic is

$$\frac{(-1)^n}{n!}\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{\partial^m}{\partial \xi^m} \pm \frac{\partial^m}{\partial \eta^m}\right)\frac{1}{r} \equiv (-1)^{n-m}\frac{(2m)!}{2^{2m}m!\,n!}(\xi^m \pm \eta^m)\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r^{2m+1}}.$$

The differential $\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r^{2m+1}}$ may be calculated by means of the well-known formula

$$\frac{d^p}{dz^p} F(z^2) = (2z)^p F^p(z^2) + \frac{p(p-1)}{1}(2z)^{p-2} F^{(p-1)}(z^2)$$
$$+\frac{p(p-1)(p-2)(p-3)}{1.2}(2z)^{p-4} F^{(p-2)}(z^2) + \dots;$$

putting $\qquad F(z^2) = \dfrac{1}{(x^2+y^2+z^2)^{m+\frac{1}{2}}}, \quad p = n-m,$

we have

$$\frac{\partial^{n-m}}{\partial z^{n-m}} \frac{1}{r^{2m+1}} = (2z)^{n-m}(-1)^{n-m}\left(\frac{2m+1}{2}\right)\left(\frac{2m+3}{2}\right)\dots\left(\frac{2n-1}{2}\right)\frac{1}{r^{2n+1}}$$
$$+(2z)^{n-m-2}\frac{(n-m)(n-m-1)}{1}(-1)^{n-m-1}\left(\frac{2m+1}{2}\right)\dots\left(\frac{2n-3}{2}\right)\frac{1}{r^{2n-1}}$$
$$+(2z)^{n-m-4}\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{1.2}$$
$$+\dots; \qquad \times(-1)^{n-m-2}\left(\frac{2n+1}{2}\right)\dots\left(\frac{2n-5}{2}\right)\frac{1}{r^{2n-3}}$$

the harmonic is therefore

$$\frac{(2m)!}{2^{2m-1}\,n!\,m!}\,{\cos \atop \sin}\, m\phi\,\frac{1}{r^{n+1}}(2m+1)(2m+3)\dots(2n-1)$$
$$\times \nu^m\left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \dots\right\},$$

where $\mu = \cos\theta$, $\nu = \sin\theta$; this is equal to

$$\frac{(n-m)!}{n!\,2^{m-1}}\,{\cos \atop \sin}\, m\phi\,\frac{1}{r^{n+1}}\,T_n^m(\mu),$$

according to the notation in Ferrers' *Spherical Harmonics* (p. 77). Maxwell considers these harmonics as the potentials due to certain singular points placed at the origin; from this point of view, these harmonics may be designated as *point-harmonics.*

The System of Line-Harmonics.

4. Next consider the harmonics derived in a similar manner from

$$\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}.$$

If matter of line-density $\frac{1}{2}$ be distributed along the positive part of the axis of z, from $z=0$ to $z=\infty$, and matter of line-density $-\frac{1}{2}$ be distributed along the negative part of that axis, the potential at any point due to this system is $\int_0^z \frac{1}{r}\,dz$, which is equal to $\log\sqrt{\frac{r+z}{r-z}}$. Inverting this system with regard to the origin, we see that

$$\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}$$

is the potential of a distribution of matter along the axis of z, of line density $\frac{1}{2z}$, positive matter on the positive side, and negative matter on the negative side of the origin. I propose to call such a line distribution a *singular line* of zero degree and unit strength. This line corresponds to Maxwell's singular point of zero degree.

A singular line of the first degree consists of two parallel lines of zero degree of infinite strength, the line joining corresponding points being in any given direction h_1, and the product of the strength into the distance between corresponding points being finite and measuring the strength of the singular line of the first degree; the direction of the line joining corresponding points is the axis of the corresponding harmonic. In a similar manner a singular line of the second degree is formed from two parallel singular lines of the first degree; proceeding thus, we have the conception of a singular line of any degree n, with n axes in arbitrary directions. We thus derive a harmonic with any n given poles $h_1, h_2 \dots h_n$. The potential due to this singular line is

$$\frac{(-1)^n}{n!}\,\frac{d^n}{dh_1\,dh_2\dots dh_n}\left(\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right);$$

this may be called the line harmonic with axes $h_1, h_2 \dots h_n$.

When all the poles coincide with the axis of z, we have

$$\frac{Q_n(\cos\theta)}{r^{n+1}}=\frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left\{\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right\},$$

$Q_n(\cos\theta)$ denoting the zonal surface harmonic, or Legendre's coefficient of the second kind; we have thus, for this zonal harmonic, the expression

$$Q_n(\cos\theta)=\frac{(-1)^n r^{n+1}}{n!}\,\frac{d^n}{dz^n}\left\{\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right\}$$

$$=\frac{(-1)^n r^{n+1}}{n!}\,\frac{d^n}{dz^n}\left\{\frac{1}{r}\log\frac{r+z}{\sqrt{r^2-z^2}}\right\}\quad\dots\dots\dots(6),$$

which is analogous to the well-known formula (5) for the zonal harmonic of the first kind.

Since $\frac{1}{r}$ and $\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}$ are the only harmonics of degree -1 which are functions of r and θ only, P_n and Q_n are the only zonal surface harmonics of degree n.

Carrying out the differentiation in (6), by means of Leibnitz's theorem, and remembering that

$$\frac{\partial}{\partial z}\log\sqrt{\frac{r+z}{r-z}} = \frac{1}{r},$$

we obtain the formula

$$Q_n = P_n\log\sqrt{\frac{1+\mu}{1-\mu}} - \tfrac{1}{2}\left(P_{n-1}P_0 + \tfrac{1}{2}P_{n-2}P_1 + \tfrac{1}{3}P_{n-3}P_2 + \dots\right).$$

This expression for Legendre's coefficient of the second kind is, I believe, new; the ordinary formula obtained for it from the differential equation is

$$Q_n = P_n\log\sqrt{\frac{1+\mu}{1-\mu}} - \left(\frac{2n-1}{1\,.\,n}P_{n-1} + \frac{2n-5}{2\,.\,n-1}P_{n-3} + \frac{2n-9}{5\,.\,n-2}P_{n-5} + \dots\right).$$

We thus obtain the curious identity

$$P_{n-1}P_0 + \tfrac{1}{2}P_{n-2}P_1 + \tfrac{1}{3}P_{n-3}P_2 + \dots = 2\left(\frac{2n-1}{1\,.\,n}P_{n-1} + \frac{2n-5}{2\,.\,n-1}P_{n-3} + \dots\right),$$

which could be verified independently.

From the formula (6) it appears that $Q_n(\cos\theta)$ may be defined as the coefficient of h^n in the expansion, in powers of h, of

$$\frac{1}{\sqrt{1-2h\mu+h^2}}\log\frac{\sqrt{1-2\mu h+h^2}+(\mu-h)}{\sqrt{1-\mu^2}}.$$

The line harmonic with $n-m$ poles coincident with the axis of z, and the remaining m poles arranged symmetrically in the equatorial plane, is

$$\frac{(-1)^n}{n!}\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{\partial^m}{\partial\xi^m} \pm \frac{\partial^m}{\partial\eta^m}\right)\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right],$$

or

$$\frac{(-1)^n}{2^m n!}\left(\xi^m \pm \eta^m\right)\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{d}{r\,dr}\right)^m\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right],$$

which is equal to

$$\frac{(-1)^n}{2^{m-1}n!}\,r^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{d}{r\,dr}\right)^m\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right] \text{ ....(7)},$$

where $\frac{d}{dr}$ denotes differentiation with respect to r, z being kept constant; this expression, multiplied by r^{n+1}, gives the tesseral harmonics of the second kind, $Q_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi$; thus

$$Q_n^m = C\,(-1)^n\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{d}{r\,dr}\right)^m\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right],$$

C denoting a numerical factor.

The formula (7) may be written

$$\frac{(-1)^m\,2^{n-2m+1}\,(n-m)!}{n!}\,r^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,\frac{d^m}{(r\,dr)^m}\,\frac{Q_{n-m}(z)}{r^{n-m+1}}.$$

Formulæ for the Tesseral Harmonics of the first and second kind.

5. The tesseral harmonics of the first and second kind can, as we have shown in section 1, be derived by differentiating certain harmonics of degree -1, with respect to z only. The harmonics of degree -1 which contain $\begin{matrix}\cos\\ \sin\end{matrix} m\phi$ as factors, and no other factor involving ϕ, are

$$\frac{1}{r}\,\frac{(r+z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

and

$$\frac{1}{r}\,\frac{(r-z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\begin{matrix}\cos\\ \sin\end{matrix} m\phi;$$

from these we obtain the harmonics

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi\,.\,\frac{1}{(x^2+y^2)^{\frac{1}{2}m}}\,\frac{\partial^n}{\partial z^n}\,\frac{(r+z)^m}{r} \quad\text{and}\quad \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,.\,\frac{1}{(x^2+y^2)^{\frac{1}{2}m}}\,\frac{\partial_n}{\partial z^n}\,\frac{(r-z)^n}{r},$$

which may be written

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi\,.\,\frac{1}{(x^2+y^2)^{\frac{1}{2}m}}\,\frac{\partial^{n+1}}{\partial z^{n+1}}\,(r\pm z)^m \text{ .... (8)}.$$

When n is less than m, these are the two distinct harmonics of type

$$\begin{matrix}\sin\\ \cos\end{matrix} m\phi\,.\,H_n\left\{z,\ \sqrt{x^2+y^2}\right\}.$$

but if $n > m$, and m is integral, these harmonics are not distinct, because the terms in $(r \pm z)^m$ which contain even powers of r, vanish when differentiated $n+1$ times with respect to z.

When $n > m$, either of the expressions

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \cdot \frac{1}{(x^2+y^2)^{\frac{1}{2}m}} \frac{\partial^n}{\partial z^n} \frac{(r \pm z)^m}{r}$$

gives the tesseral harmonics of the first kind. Those of the second kind are

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \cdot \frac{1}{(x^2+y^2)^{\frac{1}{2}m}} \frac{\partial^n}{\partial z^n} \left[\frac{1}{r} \left\{(z+r)^m + (z-r)^m\right\} \log \sqrt{\frac{r+z}{r-z}}\right] \quad \ldots (9),$$

which may also be written

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \cdot \frac{1}{(x^2+y^2)^{\frac{1}{2}m}} \frac{\partial^{n+1}}{\partial z^{n+1}} \left[\left\{(z+r)^m - (z-r)^m\right\} \log \sqrt{\frac{r+z}{r-z}}\right].$$

These are obtained by differentiating, with respect to z, the two harmonics

$$\log \sqrt{\frac{r+z}{r-z}} \cos m\phi \left\{\frac{(z+r)^m + (z-r)^m}{r\,(x^2+y^2)^{\frac{1}{2}m}}\right\}$$

$$-\tan^{-1}\frac{y}{x} \sin m\phi \left\{\frac{(z+r)^m - (z-r)^m}{r\,(x^2+y^2)^{\frac{1}{2}m}}\right\},$$

and

$$\log \sqrt{\frac{r+z}{r-z}} \sin m\phi \left\{\frac{(z+r)^m + (z-r)^m}{r\,(x^2+y^2)^{\frac{1}{2}m}}\right\}$$

$$+\tan^{-1}\frac{y}{x} \cos m\phi \left\{\frac{(z+r)^m - (z-r)^m}{r\,(x^2+y^2)^{\frac{1}{2}m}}\right\},$$

for when $n > m$, (m integral), the coefficients of

$$\cos m\phi \,.\, \tan^{-1}\frac{y}{x}, \quad \sin m\phi \,.\, \tan^{-1}\frac{y}{x},$$

vanish on differentiation n times with respect to z. If $n < m$, the harmonics obtained from these, contain ϕ as a factor, and in that case, both harmonics of the type

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \,.\, H_n \left\{z, \sqrt{x^2+y^2}\right\},$$

are given by (8).

The formula (8) may be written

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \, . \, (x^2+y^2)^{\frac{1}{2}m} \frac{\partial^n}{\partial z^n} \left\{ \frac{1}{r\,(r \pm z)^m} \right\};$$

thus, when $n > m$, the tesseral surface harmonic is

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \, . \, \sin^m \theta \, . \, P_n^m (\cos\theta),$$

where $$P_n^m (\cos\theta) = A r^{m+n+1} \frac{\partial^n}{\partial z^n} \frac{1}{r\,(r \pm z)^m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

A denoting a constant factor. This formula (10) is analogous to the formula (5) which gives $P_n (\cos\theta)$; the factor A depends upon the exact definition of the associated function $P_n^m (\cos\theta)$.

In Jacobi's formula

$$\frac{d^{m-1}(1-t^2)^{m-\frac{1}{2}}}{dt^{m-1}} = (-1)^{m-1} \frac{1\,.\,3\,.\,5\ldots(2m-1)}{m} \sin m\psi,$$

where $t = \cos\psi$, putting $t = \frac{\iota z}{\sqrt{x^2+y^2}}$, we have

$$\frac{\partial^{m-1} r^{2m-1}}{(x^2+y^2)^{\frac{1}{2}m}\,\partial z^{m-1}} = \frac{1\,.\,3\,.\,5\ldots(2m-1)}{2m} \left\{ \cot^m \tfrac{1}{2}\theta - (-\tan \tfrac{1}{2}\theta)^m \right\};$$

or $$\frac{1}{(x^2+y^2)^{\frac{1}{2}m}} \, . \, \frac{\partial^{m-1} r^{2m-1}}{\partial z^{m-1}} = \frac{1\,.\,3\ldots(2m-1)}{2m} \, . \, \frac{(z+r)^m + (z-r)^m}{(x^2+y^2)^{\frac{1}{2}m}},$$

hence when $n > m$, the expressions (8) are equal (disregarding a constant factor) to

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \frac{1}{(x^2+y^2)^{\frac{1}{2}m}} \frac{\partial^{n+m}}{\partial z^{n+m}} r^{2m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The expression (11) is therefore another expression for the tesseral harmonics of the first kind; it should be observed that it is obtained by changing m into $-m$ in the expression

$$\begin{matrix}\cos\\ \sin\end{matrix} m\phi \, . \, (x^2+y^2)^{\frac{1}{2}m} \frac{\partial^{n-m}}{\partial z^{n-m}} \frac{1}{r^{2m+1}},$$

in section (3). We have

$$P_n^m (\cos\theta) = B \, . \, r^{n+m+1} \frac{\partial^{n-m}}{\partial z^{n-m}} \frac{1}{r^{2m+1}},$$

when B is a constant factor, as an expression for the associated function $P_n^m (\cos\theta)$.

We have

$$\frac{\partial^{n+m}}{\partial z^{n+m}}(x^2+y^2+z^2)^{m-\frac{1}{2}} = (2z)^{n+m}\,(m-\tfrac{1}{2})(m-\tfrac{3}{2})\,\dots\,(-n+\tfrac{1}{2})\,r^{-2n-1}$$

$$+\frac{(n+m)(n+m-1)}{1}\,(2z)^{n+m-2}\,(m-\tfrac{1}{2})\,\dots\,(-n+\tfrac{3}{2})\,r^{-2n+1}$$

$$+\frac{(n+m)(n+m-1)(n+m-2)(n+m-3)}{1\,.\,2}\,(2z)^{n+m-4}$$

$$+\dots\dots\dots\dots \qquad \times(m-\tfrac{1}{2})\,\dots\,(-n+\tfrac{5}{2})\,r^{-2n+3}$$

$$=\frac{2^{n+m}}{r^{n-m+1}}(m-\tfrac{1}{2})(m-\tfrac{3}{2})\dots(-n+\tfrac{1}{2})\Big[\mu^{n+m}-\frac{(n+m)(n+m-1)}{1\,.\,n-\frac{1}{2}}\cdot\frac{1}{2^2}\mu^{n+m-2}$$

$$+\frac{(n+m)(n+m-1)(n+m-2)(n+m-3)}{1\,.\,2\,(n-\frac{1}{2})(n-\frac{3}{2})}\cdot\frac{1}{2^4}\,\mu^{n+m-4}-\dots\Big],$$

and this must be equal to $A\,.\,\dfrac{r^m}{r^{n+1}}(1-\mu^2)^n\,T_n^m$, where A is a numerical quantity, or* to

$$\frac{A}{r^{n-m+1}}\,(1-\mu^2)\,\frac{2n\,(2n-1)\,\dots\,(n-m+1)}{2^n\,n\,!}\,\{\mu^{n-m}-\dots\};$$

hence
$$A=\frac{2^{n+m}\,(m-\frac{1}{2})\,\dots\,(-n+\frac{1}{2})}{(-1)^m\,\dfrac{2n\,(2n-1)\,\dots\,(n-m+1)}{2^n\,n\,!}}.$$

The expression (11) gives, therefore, for $T_n^m\,(\mu)$, the formula

$$T_n^m\,(\cos\theta)=(-1)^m\,\frac{(2n)\,!}{(n-m)\,!\;n\,!\;(m-\frac{1}{2})\,\dots\,(-n+\frac{1}{2})\,2^{2n+m}}$$

$$\times\,\frac{r^{n+1}}{(x^2+y^2)^{\frac{1}{2}m}}\,\frac{\partial^{n+m}}{\partial z^{n+m}}\,r^{2m-1}\quad\dots\dots\dots\dots\dots\dots\ (12).$$

It is stated by Thomson and Tait (*Natural Philosophy*, p. 174), that

$$\frac{\partial^{n+m}\,(r^{2m-1})\;\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi}{\partial z^{n+m}\quad(x^2+y^2)^{\frac{1}{2}m}}$$

and
$$\left(\frac{\partial}{\partial z}\right)^{n+m+1}\left[r^{2m+1}\left(\frac{d}{rdr}\right)^m\left(\frac{1}{r}\log\frac{r+z}{r-z}\right)\right](x^2+y^2)^{\frac{1}{2}m}\,\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi$$

are, except when $m=0$, the two distinct harmonics of the type

$$H_n\left\{z,\ \sqrt{x^2+y^2}\right\}\,\begin{matrix}\sin\\ \cos\end{matrix}\,m\phi\,;$$

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* See Ferrers' *Spherical Harmonics*, p. 76.

this is, however, not correct, for the second expression is really always identical (except for a numerical factor) with the first. It is, in fact, easily seen that, in the second expression, the quantity $\log \frac{r+z}{r-z}$ disappears on differentiation, which we know should not be the case. It is easy to verify in simple cases ($m = 1, 2, 3$) that

$$\frac{\partial}{\partial z}\left[r^{2m+1}\left(\frac{d}{r\,dr}\right)^m\left(\frac{1}{r}\log\frac{r+z}{r-z}\right)\right](x^2+y^2)^{\frac{1}{2}m}\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi$$

is equal to
$$C\cdot\frac{r^{2m-1}}{(x^2+y^2)^{\frac{1}{2}m}}\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi,$$

so that the two harmonics of degree zero, given by Thomson and Tait (p. 173), are identical.

The second harmonic should be

$$r^m\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\,\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{d}{r\,dr}\right)^m\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right],$$

or the equivalent form

$$\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\cdot\frac{1}{(x^2+y^2)^{\frac{1}{2}m}}\frac{\partial^n}{\partial z^n}\left[\frac{1}{r}\left\{(z+r)^m+(z-r)^m\right\}\log\sqrt{\frac{r+z}{r-z}}\right];$$

this does not become identical with the first harmonic when $m = 0$, but becomes

$$\frac{\partial^n}{\partial z^n}\left[\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}\right],$$

the zonal harmonic of the second kind, of degree $-n-1$.

Circulatory Harmonics.

6. If a sheet of magnetic matter of constant strength is distributed on the half of the plane xz, for which x is negative, its potential is $\tan^{-1}\frac{y}{x}$; inverting this system with respect to the origin, we obtain the expression $\frac{1}{r}\tan^{-1}\frac{y}{x}$ for the potential of a sheet of magnetic matter of strength inversely proportional to the distance from the origin; we may call such a sheet a singular sheet of zero degree. A singular sheet of any degree n will then be obtained, as in the case of singular points and lines, by displacing the sheet in the directions of any n axes arbitrarily chosen; the potential of such a singular sheet is then

$$\frac{d^n}{dh_1\,dh_2\ldots dh_n}\left\{\frac{1}{r}\tan^{-1}\frac{y}{x}\right\}.$$

These harmonics I propose to call "circulatory harmonics," on account of the presence in them of the quantity $\tan^{-1}\frac{y}{x}$. They might also be called "sheet harmonics."

If all the axes coincide with the axis of z, we obtain the harmonic

$$\frac{P_n(\cos\theta)}{r^{n+1}}\,.\,\phi,$$

which may be called the zonal circulatory harmonic.

A system of tesseral harmonics of the form

$$\frac{1}{r^{n+1}}\,\frac{\cos}{\sin}\,m\phi\,(\phi\,.\,T_n^m+V_n^m),$$

where T_n^m has its usual meaning, is obtained by making $n-m$ axes coincide with the axis of z, the other m axes being arranged symmetrically in the equatorial plane; this expression is obtained by carrying out the differentiations in the expression

$$\left(\frac{\partial}{\partial z}\right)^{n-m}\left\{\left(\frac{\partial}{\partial \xi}\right)^{m}\pm\left(\frac{\partial}{\partial \eta}\right)^{m}\right\}\left[\frac{1}{r}(\log\xi-\log\eta)\right].$$

A system of harmonics which may be called circulatory harmonics of the second kind, is given by the expression

$$\frac{d^n}{dh_1\,dh_2\,\dots\,dh_n}\left\{\frac{1}{r}\tan^{-1}\frac{y}{x}\,.\,\log\sqrt{\frac{r+z}{r-z}}\right\}.$$

The zonal harmonic of this system is $\dfrac{Q_n(\cos\theta)}{r^{n+1}}\,.\,\phi$.

Cylindrical Harmonics.

7. If, in the equation $\nabla^2 V=0$, we put $V=e^{kz}U$, where U is a function of x and y only, we have a system of harmonics $e^{kz}U$, where U is given by the differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+k^2U=0\;\dots\dots\dots\dots\dots\dots\dots\; (13).$$

That solution of this equation which is symmetrical with respect to the axis is $\quad AJ_0\left(k\sqrt{x^2+y^2}\right)+BY_0\left(k\sqrt{x^2+y^2}\right),$

where J_0 and Y_0 denote the two Bessel's functions of zero order, thus

$$e^{kz}J_0\left(k\sqrt{x^2+y^2}\right),\quad e^{kz}Y_0\left(k\sqrt{x^2+y^2}\right),$$

are the two cylindrical harmonics which are symmetrical with respect to the z axis. Putting, for simplicity, $k=1$, further solutions of the equation (13) may be found by differentiating J_0, Υ_0 with respect to axes in the xy plane; thus we have harmonics of the type

$$\epsilon^z \frac{d^m}{dh_1\, dh_2 \ldots dh_m} J_0(\sqrt{x^2+y^2}), \quad \epsilon^z \frac{d^m}{dh_1\, dh_2 \ldots dh_m} \Upsilon_0(\sqrt{x^2+y^2}).$$

If the axes are arranged symmetrically, we obtain the harmonics

$$\epsilon^z \left(\frac{\partial^m}{\partial \xi^m} \pm \frac{\partial^m}{\partial \eta^m}\right) J_0(\sqrt{\xi\eta}), \quad \epsilon^z \left(\frac{\partial^m}{\partial \xi^m} \pm \frac{\partial^m}{\partial \eta^m}\right) \Upsilon_0(\sqrt{\xi\eta}).$$

We have
$$\frac{\partial^m}{\partial \xi^m} J_0(\sqrt{\xi\eta}) = \eta^m \frac{d^m}{d(\xi\eta)^m} J_0(\sqrt{\xi\eta});$$

thus, writing $\rho^2 = \xi\eta$, we obtain the harmonics

$$(\xi^m \pm \eta^m) \frac{d^m}{d(\rho^2)^m} J_0(\rho) \quad \text{or} \quad \rho^m \frac{\cos}{\sin} m\phi \frac{d^m}{d(\rho^2)^m} J_0(\rho):$$

writing with the usual notation

$$(-2\rho)^m \frac{d^m}{d(\rho^2)^m} J_0(\rho) = J_m(\rho),$$

we have the harmonics $\frac{\cos}{\sin} m\phi \,.\, J_m(\rho)$. In a similar manner we obtain the corresponding harmonics $\frac{\cos}{\sin} m\phi \,.\, \Upsilon_m(\rho)$ of the second kind, where

$$\Upsilon_m(\rho) = (-2\rho)^m \frac{d^m}{d(\rho^2)^m} \Upsilon_0(\rho).$$

We are thus led naturally to the Bessel's functions of order m, by the process of differentiation with respect to m axes symmetrically arranged in the xy plane.

It is obvious, if we write the equation (13) in the form

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + V = 0,$$

that
$$\phi \,.\, J_0(\rho) = \tan^{-1}\frac{y}{x} \,.\, J_0(\sqrt{x^2+y^2}),$$

and
$$\phi \,.\, \Upsilon_0(\rho) = \tan^{-1}\frac{y}{x} \,.\, \Upsilon_0(\sqrt{x^2+y^2}),$$

also satisfy the equation: we obtain thus the two harmonics

$$\epsilon^z . \tan^{-1}\frac{y}{x} \, \frac{J_0}{\Upsilon_0}(\sqrt{x^2+y^2}),$$

which may be called the circulatory cylindrical harmonics of zero order.

The corresponding harmonics of order m are

$$e^z\left(\frac{\partial^m}{\partial\xi^m}\pm\frac{\partial^m}{\partial\eta^m}\right)\left[(\log\xi-\log\eta)\,\genfrac{}{}{0pt}{}{J_0}{Y_0}\,(\sqrt{\xi\eta})\right].$$

Carrying out the differentiations, we find for the harmonics of the first kind

$$\phi\,.\cos m\phi\,.\,J_m(\rho)+\sin m\phi\,.\,U_m,$$

$$\phi\,.\sin m\phi\,.\,J_m(\rho)-\cos m\phi\,.\,U_m,$$

where $$U_m=m\,.\,\frac{1}{\rho}\,J_{m-1}(\rho)+\frac{m(m-1)}{2}\,.\,\frac{2}{\rho^2}\,J_{m-2}(\rho)$$

$$+\frac{m(m-1)(m-2)}{3}\,.\,\frac{2^2}{\rho^3}\,J_{m-3}(\rho)+\dots\,.$$

For the harmonics of the second kind, Y must be written for J in these expressions. These are the circulatory cylindrical harmonics of order m, of the first and second kinds respectively.

The most general Spherical Harmonics of given order.

8. Writing the equation $\nabla^2 V=0$ in the form

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\left(\frac{\partial}{\partial z}\right)^2V=0,$$

we see that the most general solution which is symmetrical with respect to the z axis is

$$V=J_0\left(\sqrt{x^2+y^2}\,\frac{d}{dz}\right)f(z)+Y_0\left(\sqrt{x^2+y^2}\,\frac{d}{dz}\right)\phi(z),$$

where $f(z)$ and $\phi(z)$ are arbitrary functions of z, being constants with respect to x and y. Those solutions of the equation which contain $\genfrac{}{}{0pt}{}{\cos}{\sin}\,m\phi$ as factors are

$$\genfrac{}{}{0pt}{}{\cos}{\sin}\,m\phi\,.\,J_m\left(\sqrt{x^2+y^2}\,\frac{d}{dz}\right)f(z),\quad \genfrac{}{}{0pt}{}{\cos}{\sin}\,m\phi\,.\,Y_m\left(\sqrt{x^2+y^2}\,\frac{d}{dz}\right)\phi(z).$$

Now we know* that $J_m(\rho)$, $Y_m(\rho)$ are expressed as definite integrals thus :—

$$J_m(\rho)=\frac{\rho^m}{\pi}\int_0^\pi\sin^{2n}\psi\,.\,e^{\iota\rho\cos\psi}\,d\psi,$$

$$Y_m(\rho)=\rho^m\int_0^\infty\sinh^{2m}u\,.\,e^{\iota\rho\cosh u}\,du.$$

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* See Heine's *Kugelfunctionen*, Vol. I., p. 234.

We thus obtain as an expression for the most general spheric harmonic which is independent of ϕ, the expression

$$\int_0^\pi e^{\iota\cos\psi\sqrt{x^2+y^2}\frac{d}{dz}} f(z)\,d\psi + \int_0^\infty e^{\iota\cosh u\sqrt{x^2+y^2}\frac{d}{dz}} \phi(z)\,d\psi,$$

which is equal to

$$\int_0^\pi f(z+\iota\sqrt{x^2+y^2}\cos\psi)\,d\psi + \int_0^\infty \phi(z+\iota\sqrt{x^2+y^2}\cosh u)\,du \quad \text{...(14}$$

where f and φ are arbitrary functions. The first integral is wel known.

This expression is the most general spherical harmonic of order irrespective of degree. It should be observed that the imagina part of the second integral is the same as the real part of the first.

The most general spherical harmonics of order m are

$$\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{1}{2}m}\int_0^\pi \sin^{2m}\psi\,.\,e^{\iota\cos\psi\sqrt{x^2+y^2}\frac{d}{dz}} f(z)\,d\psi$$

$$+\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{1}{2}m}\int_0^\infty \sinh^{2m}u\,.\,e^{\iota\cosh u\sqrt{x^2+y^2}\frac{d}{dz}} \phi(z)\,du,$$

or

$$\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{1}{2}m}\int_0^\pi \sin^{2m}\psi\,.\,f(z+\iota\sqrt{x^2+y^2}\cos\psi)\,d\psi$$

$$+\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{1}{2}m}\int_0^\infty \sinh^{2m}u\,.\,\phi(z+\iota\sqrt{x^2+y^2}\cosh u)\,du \quad \text{.........(15}$$

If we use the expansions of J_0, J_m in powers of their arguments, v find for the most general harmonic of the first kind, of zero order,

$$f(z)-\frac{x^2+y^2}{2^2}f''(z)+\frac{(x^2+y^2)^2}{2^2\,.\,4^2}f^{\text{iv}}(z)-\ldots \quad \text{.........(16}$$

and for the most general harmonic of the first kind, of order m,

$$\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{1}{2}m}$$

$$\times\left\{f(z)-\frac{x^2+y^2}{2\,(2m+2)}f''(z)+\frac{(x^2+y^2)^2}{2\,.\,4\,(2m+2)(2m+4)}f^{\text{iv}}(z)-\ldots\right\} \quad \text{.........(17}$$

when $f(z)$ is an arbitrary function of z.

If we put $$f(z)=z^{n-m},$$

we obtain the ordinary harmonics of degree n; writing

$$x \text{ [illegible] } + y^2 = r^2\nu^2$$

the formulæ give

$$r^n\left[\mu^n-\frac{n(n-1)}{2^2}\mu^{n-2}\nu^2+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}\mu^{n-4}\nu^4-\ldots\right],$$

which is $r^n.P_n(\mu)$, and

$$\begin{matrix}\cos\\ \sin\end{matrix}m\phi\,.\,r^n\,.\,\nu^m$$

$$\times\left[\mu^m-\frac{(n-m)(n-m-1)}{2(2m+2)}\mu^{m-2}\nu^2\right.$$
$$\left.+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4(2m+2)(2m+4)}\mu^{m-4}\nu^4-\ldots\right],$$

which is* $r^m\begin{matrix}\cos\\ \sin\end{matrix}m\phi\,.\,T_n^m(\mu)$.

On certain Properties of Symmetric, Skew Symmetric, and Orthogonal Matrices.

By HENRY TABER, Clark University, Mass., U.S.A.

[*Read June* 11*th*, 1891.]

§ 1. *Introductory.*

In the *Messenger of Mathematics* (2), Vol. XIV., the late A. Buchheim has employed a method of proof of the reality of the latent roots of a symmetric matrix which may be extended to prove certain important theorems relating to symmetric, skew symmetric, and orthogonal matrices. Buchheim's proof is virtually an extension of the process employed by Tait in his "Elements of Quaternions" (Chap. v.) for the proof of the reality of the latent roots of a symmetric matrix of the third order. I shall give by this method a proof of the well-known property of a real symmetric matrix ϕ, that $\phi-g$, for a multiple latent root g of ϕ, has a nullity equal to its vacuity (which is equal to the multiplicity of g);[*] and I shall show that this property is characteristic likewise of real skew symmetric and of real orthogonal matrices. I shall also show by this means that the latent roots of a real skew symmetric matrix that are other than zero are purely imaginary.

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* See Thomson and Tait, p. 188.

Consider the two sets of variables $(x_1, x_2, \dots x_\omega)$ and $(y_1, y_2, \dots y_\omega)$, which may be regarded as the co-ordinates of points in ω-way space, and let

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1\omega}x_\omega, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2\omega}x_\omega, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ y_\omega &= a_{\omega 1}x_1 + a_{\omega 2}x_2 + \dots + a_{\omega\omega}x_\omega; \end{aligned}$$

these ω equations may be written in Cayley's "abbreviated notation" as

$$(y_1, y_2, \dots y_\omega) = \left(\begin{array}{|cccc|} a_{11} & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega} \end{array} \right\langle x_1, x_2, \dots x_\omega),\text{[†]}$$

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[*] The matrix $\phi - g$ is obtained from ϕ by subtracting g from all the constituents in the principal diagonal of ϕ. The terms *vacuity* and *nullity* are Sylvester's (*American Jour. of Math.*, Vol. VI.). If ϕ is the matrix,

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega}; \end{matrix}$$

and, if all the minors of this matrix of order $\omega - i + 1$ vanish, but not all the minors of order $\omega - i$, its nullity is of order i; if $i = 1$, ϕ is said to have simple nullity. The term *vacuity* has reference to the number of roots equal to zero of the equation in g,

$$\begin{vmatrix} a_{11} - g & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} - g & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{2\omega} & \dots & a_{\omega\omega} - g \end{vmatrix} = 0.$$

If i roots of this equation are zero, the vacuity of ϕ is i; if $i = 1$, ϕ is said to be simply vacuous. The roots of this equation are termed by Sylvester *latent roots* of ϕ. If the vacuity of ϕ is i, the sum of all the principal r^{th} minors of ϕ is zero, for $r = 0, 1, 2, \dots i-1$, but the sum of the principal i^{th} minors is not zero; therefore the nullity of ϕ cannot exceed its vacuity. Si[illegible]nt roots of $\phi - g$ are severally equal to the several latent roots of ϕ le[illegible] m-tuple latent root of ϕ, the vacuity of $\phi - g$ is m.

[†] "Memoir on the Theory of Matrices," *Phil.*[illegible]

or, if ϕ denote the above square array or matrix, as

$$(y_1, y_2, \dots y_\omega) = (\phi \between x_1, x_2, \dots x_\omega).$$

If we put $$(\phi \between x_1, x_2, \dots x_\omega) = (g \between x_1, x_2, \dots x_\omega),$$

where g is not a general symbol of operation, but a scalar considered as involving the matrix unity—*i.e.*, if

$$a_{r1}x_1 + a_{r2}x_2 + \dots + a_{r\omega}x_\omega = gx_r,$$

for $r = 1, 2, \dots \omega$—the condition necessary and sufficient that this system of ω scalar equations shall have a solution other than all the x's equal to zero is

$$\begin{vmatrix} a_{11}-g & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22}-g & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega}-g \end{vmatrix} = 0.$$

Following Sylvester, the roots of this equation will be termed *latent roots* of ϕ; and the set $(x_1, x_2, \dots x_\omega)$, for which

$$(\phi \between x_1, x_2, \dots x_\omega) = (g \between x_1, x_2, \dots x_\omega),$$

will be termed an *axial set* to ϕ corresponding to the latent root g.

A latent root of ϕ and its corresponding axial set may be imaginary. Separating the real and imaginary parts of each, we have

$$(\phi \between x_1 + y_1\sqrt{-1},\ x_2 + y_2\sqrt{-1}, \dots x_\omega + y_\omega\sqrt{-1})$$
$$= (g + h\sqrt{-1} \between x_1 + y_1\sqrt{-1}, x_2 + y_2\sqrt{-1}, \dots x_\omega + y_\omega\sqrt{-1}),$$

where g and h and the x's and y's are real. If ϕ is real, equating separately the real and imaginary parts of the ω scalar equations represented above in Cayley's matrical notation, there result 2ω equations which are represented in the matrical notation by

$$\left.\begin{aligned} (\phi \between x_1, x_2, \dots x_\omega) &= (gx_1 - hy_1,\ gx_2 - hy_2, \dots gx_\omega - hy_\omega) \\ (\phi \between y_1, y_2, \dots y_\omega) &= (hx_1 + gy_1,\ hx_2 + gy_2, \dots hx_\omega + gy_\omega) \end{aligned}\right\} \dots (A);$$

whence follows

$$\phi\,(x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1})$$
$$=(gx_1-hy_1-\overline{hx_1+gy_1}\sqrt{-1},\ gx_2-hy_2-\overline{hx_2+gy_2}\sqrt{-1},\ \dots$$
$$\dots\ gx_n-hy_n-\overline{hx_n+gy_n}\sqrt{-1})$$
$$=(g-h\sqrt{-1})\!(x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1}).$$

It is convenient to regard two sets $(\xi_1, \xi_2, \dots \xi_n)$ and $(\eta_1, \eta_2, \dots \eta_n)$ as susceptible of addition, so that

$$(\xi_1, \xi_2, \dots \xi_n)+(\eta_1, \eta_2, \dots \eta_n)=(\xi_1+\eta_1, \xi_2+\eta_2, \dots \xi_n+\eta_n);$$

the matrix ϕ must then be a distributive operator, thus

$$(\phi)[(\xi_1, \xi_2, \dots \xi_n)+(\eta_1, \eta_2, \dots \eta_n)]=(\phi)\!(\xi_1, \xi_2, \dots \xi_n)+(\phi)\!(\eta_1, \eta_2, \dots \eta_n).$$

The conception of two sets as susceptible of addition is virtually involved in regarding matrices as susceptible of addition. The geometrical interpretation of the sum of two sets is obvious. Adopting this convention, equations (A) become

$$(\phi)\!(x_1, x_2, \dots x_n)=(g)\!(x_1, x_2, \dots x_n)-(h)\!(y_1, y_2, \dots y_n),$$
$$(\phi)\!(y_1, y_2, \dots y_n)=(h)\!(x_1, x_2, \dots x_n)+(g)\!(y_1, y_2, \dots y_n);$$

and so, as before,

$$(\phi)\!(x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1})$$
$$=(\phi)\!(x_1, x_2, \dots x_n)-(\sqrt{-1})\!(\phi)\!(y_1, y_2, \dots y_n)$$
$$=(g-h\sqrt{-1})[(x_1, x_2, \dots x_n)-(\sqrt{-1})\!(y_1, y_2, \dots y_n)]$$
$$=(g-h\sqrt{-1})\!(x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1}).$$

Therefore, if any imaginary set is axial to a real matrix ϕ and corresponds to the latent root $g+h\sqrt{-1}$, its conjugate imaginary set is axial to ϕ and corresponds to the conjugate imaginary latent root $g-h\sqrt{-1}$.

Two sets of variables $(X_1, X_2, \dots X_n)$ and $(y_1, y_2, \dots y_n)$ may be regarded as susceptible of multiplication together. What may be termed the *inner* or *scalar* product may be denoted by placing the two sets in juxtaposition, and is defined as follows:

$$(X_1, X_2, \text{[illegible]} \dots y_n)\ \text{[illegible]} + X_2y_2+\dots+X_ny_n.$$

The order of multiplication is, of course, indifferent. If

$$(X_1, X_2, \dots X_n) = (\phi \between x_1, x_2, \dots x_n),$$

then

$$\begin{aligned}(X_1, X_2, \dots X_n \between y_1, y_2, \dots y_n) &= (\phi \between x_1, x_2, \dots x_n \between y_1, y_2, \dots y_n) \\ &= \Sigma_r a_{1r} x_r \,.\, y_1 + \Sigma_r a_{2r} x_r \,.\, y_2 + \dots + \Sigma_r a_{nr} x_r \,.\, y_n .\end{aligned}$$

If $\check{\phi}$ denote the *converse* or *transverse* of ϕ—*i.e.*, the matrix obtained by interchanging the rows and columns of ϕ—it is obvious that

$$(\phi \between x_1, x_2, \dots x_n \between y_1, y_2, \dots y_n) = (\check{\phi} \between y_1, y_2, \dots y_n \between x_1, x_2, \dots x_n).\text{*}$$

§ 2. *Latent Roots of a Real Skew Symmetric Matrix.*

It may now, by means of the last equation, be very simply proved that any non-zero latent root of a real skew symmetric matrix is purely imaginary. By definition, if ϕ is a skew symmetric matrix,

$$\check{\phi} = -\phi,$$

i.e., if $(\phi)_{rs}$ denote the constituent of ϕ in the r^{th} row and s^{th} column, since by definition

$$(\check{\phi})_{rs} = (\phi)_{sr},$$

therefore

$$(\check{\phi})_{rs} = (\phi)_{sr} = -(\phi)_{rs}.$$

If the set $(x_1+y_1\sqrt{-1},\ x_2+y_2\sqrt{-1},\ \dots\ x_n+y_n\sqrt{-1})$ is axial to ϕ and corresponds to the latent root $g+h\sqrt{-1}$, then

$$\begin{aligned}&(g+h\sqrt{-1})(\Sigma_r x_r^2+\Sigma_r y_r^2) \\ &= (\phi \between x_1+y_1\sqrt{-1},\ x_2+y_2\sqrt{-1},\ \dots\ x_n+y_n\sqrt{-1} \between x_1-y_1\sqrt{-1}, \\ &\qquad\qquad x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1}) \\ &= (\check{\phi} \between x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1} \between x_1+y_1\sqrt{-1}, \\ &\qquad\qquad x_2+y_2\sqrt{-1},\ \dots\ x_n+y_n\sqrt{-1}) \\ &= -(\phi \between x_1-y_1\sqrt{-1},\ x_2-y_2\sqrt{-1},\ \dots\ x_n-y_n\sqrt{-1} \between x_1+y_1\sqrt{-1}, \\ &\qquad\qquad x_2+y_2\sqrt{-1},\ \dots\ x_n+y_n\sqrt{-1}) \\ &= -(g-h\sqrt{-1})(\Sigma_r x_r^2+\Sigma_r y_r^2).\end{aligned}$$

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* See Cayley, "Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function" (*Collected Papers*, No. 153).

$$(\phi \between x_1, x_2, \dots x_n \between y_1, y_2, \dots y_n)$$

is Cayley's notation for the bipartite quadric function $\Sigma_r \Sigma_s a_{rs} x_s y_r$.

It cannot happen that $\qquad \Sigma_r x_r^2 + \Sigma_r y_r^2 = 0,$

for the equations for the determination of an axial set have a solution for each latent root: therefore $g = 0$, and every latent root of ϕ is of the form $h\sqrt{-1}$.*

§ 3. *Abbreviated Notation for Sets of Variables.*

What precedes may be written in a much more compact notation by employing a single symbol as an abbreviation for a set of ω variables. Thus for the two sets $(x_1, x_2, \dots x_\omega)$ and $(y_1, y_2, \dots y_\omega)$, we may write ρ and σ, respectively: and for the set $(\phi \between x_1, x_2, \dots x_\omega)$, we may use the more compact form $(\phi \between \rho)$ or $\phi\rho$. If the x's and the y's are the co-ordinates of two points in ω-way space, ρ and σ, respectively, may be regarded as representing the vectors from the origin to the points whose co-ordinates are severally the x's and y's; $\rho+\sigma$ will then represent the geometric sum of the vectors ρ and σ. If

$$\rho+\sigma = \pi,$$

the co-ordinates of the point at the extremity of the vector π will, of course, be x_1+y_1, x_2+y_2, &c. If $m\rho$ and $n\sigma$ are scalar multiples of the vectors ρ and σ, we have

$$\phi\,(m\rho+n\sigma) = \phi\,(m\rho)+\phi\,(n\sigma) = m\phi\rho+n\phi\sigma\,;$$

i.e., ϕ is a linear function, distributive over a sum of vectors.

The scalar or inner product of two sets ρ and σ may be denoted as before by $(\rho \between \sigma)$, but the notation $S\rho\sigma$ has among other advantages that of suggesting that the product considered is a scalar. We have

$$S\rho\sigma = x_1 y_1 + x_2 y_2 + \dots + x_\omega y_\omega = S\sigma\rho.$$

In considering the scalar product of two sets or vectors, it is convenient to regard the co-ordinate system as a rectangular unit system, when the condition

$$S\rho\sigma = 0$$

signifies that the vectors ρ and σ are normal.

As an abbreviated notation for the bipartite quadric

$$(\phi \between y_1, y_2, \dots y_\omega \between x_1, x_2, \dots x_\omega),$$

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* Prof. Cayley [illegible] "this implies that for an odd order [ϕ being real] there is always a [illegible] which is a [illegible] result."

we have $$(\phi)(\rho)(\sigma) \text{ or } S(\phi\sigma)\rho = S\rho\phi\sigma;$$

and, as before, $$S\rho\phi\sigma = S\sigma\check{\phi}\rho.$$

The demonstration of the theorem that the latent roots of a real skew symmetric matrix are of the form $h\sqrt{-1}$ may now be exhibited in the more compact notation. But first, if ϕ is real, and if

$$\phi(\rho+\sigma\sqrt{-1}) = (g+h\sqrt{-1})(\rho+\sigma\sqrt{-1}),$$

then $$\phi\rho = g\rho - h\sigma,$$

$$\phi\sigma = h\rho + g\sigma,$$

and, consequently,

$$\phi(\rho-\sigma\sqrt{-1}) = \phi\rho - \sqrt{-1}\,\phi\sigma = (g-h\sqrt{-1})(\rho-\sigma\sqrt{-1}).$$

Whence, if $$\check{\phi} = -\phi,$$

therefore $(g+h\sqrt{-1})\,S(\rho^2+\sigma^2)$

$$\begin{aligned} &= S(\rho-\sigma\sqrt{-1})\,\phi\,(\rho+\sigma\sqrt{-1}) \\ &= S(\rho+\sigma\sqrt{-1})\,\check{\phi}\,(\rho-\sigma\sqrt{-1}) \\ &= -S(\rho+\sigma\sqrt{-1})\,\phi\,(\rho-\sigma\sqrt{-1}) \\ &= -(g-h\sqrt{-1})\,S(\rho^2+\sigma^2); \end{aligned}$$

i.e., $$gS(\rho^2+\sigma^2) = 0.$$

In what follows, I shall employ the abbreviated notation for sets of quantities.

The abbreviated notation may, however, be regarded from another point of view, viz., the symbols ρ and σ may be regarded, not as mere abbreviations for sets, but explicitly as multiple quantities. Thus, let $\iota_1, \iota_2, \ldots \iota_\omega$ be ω non-scalar units whose multiplication table is given by the laws

$$\iota_r\iota_s + \iota_s\iota_r = 0,$$

for $$r \neq s, \text{ and } \iota_r^2 = -1;$$

such a system of multiple quantities has been considered by Clifford, who terms an algebra linear in these units and their products a *geometric algebra*;[*] Clifford has shown that the units ι_1, ι_2, &c. may be geometrically interpreted as constituting a normal system of unit vectors in ω-way flat space. If, then, the x's of the set of scalars $(x_1, x_2, \dots x_\omega)$ are regarded as the co-ordinates of a point referred to a rectangular unit system in ω-way flat space, we may write

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[*] "Applications of Grassmann's Extensive Algebra," "On the Classification of Geometric Algebras."

$$\iota_1 = (1, 0, \dots 0), \quad \iota_2 = (0, 1, \dots 0), \quad \dots \quad \iota_\omega = (0, 0, \dots 1),$$

and then $$\rho = (x_1, x_2, \dots x_\omega) = x_1\iota_1 + x_2\iota_2 + \dots + x_\omega\iota_\omega.$$

The symbol S must now be regarded as a selective symbol, as in quaternions selecting the scalar part of the product of two or more vectors (*i.e.*, expressions linear in the ι_1, ι_2, &c.); and, evidently then,

$$-\rho^2 = -S\rho^2 = x_1^2 + x_2^2 + \dots + x_\omega^2,$$

and if $$\sigma = y_1\iota_1 + y_2\iota_2 + \dots + y_\omega\iota_\omega,$$

$$-S\rho\sigma = x_1y_1 + x_2y_2 + \dots + x_\omega y_\omega = -S\sigma\rho.\text{*}$$

If now ϕ is employed to denote a linear vector operator, whose matrical array as an operator upon the ι's is

$$\begin{array}{|cccc|} \multicolumn{4}{c}{(\,a_{11} \quad a_{12} \quad \dots \quad a_{1\omega}\,),} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega} \end{array}$$

then
$$\begin{aligned} \phi\rho &= \phi\,(x_1\iota_1 + x_2\iota_2 + \dots + x_\omega\iota_\omega) \\ &= x_1\phi\iota_1 + x_2\phi\iota_2 + \dots + x_\omega\phi\iota_\omega \\ &= x_1\Sigma_r a_{r1}\iota_r + x_2\Sigma a_{r2}\iota_r + \dots + x_\omega\Sigma a_{r\omega}\iota_r \\ &= \Sigma_r a_{1r}x_r\,.\,\iota_1 + \Sigma_r a_{2r}x_r\,.\,\iota_2 + \dots + \Sigma_r a_{\omega r}x_r\,.\,\iota_\omega \\ &= (\phi)(x_1, x_2, \dots x_\omega)\,; \end{aligned}$$

and it is easy to see that

$$-S\rho\phi\sigma = (\phi)(y_1, y_2, \dots y_\omega)(x_1, x_2, \dots x_\omega) = -S\sigma\breve{\phi}\rho.$$

It is, however, to be observed that, if any other system of linearly independent vectors, not constituting a unit normal system, had been chosen, instead of the ι's, in terms of which other vectors were

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* The squares of the units ι_1, ι_2, &c. may equally well be put equal to $+1$, in which case $$S\rho^2 = +\Sigma_r x_r^2, \quad S\rho\sigma = +\Sigma_r x_r y_r,$$ as is the case when ρ and σ are regarded as mere abbreviations for sets.

to be represented, and if the above array defined ϕ as an operator upon this set of vectors, then we should not have

$$S\rho\phi\sigma = S\sigma\check{\phi}\rho.$$

This is equivalent to stating that the transverse of the matrical array of ϕ does not represent the conjugate linear vector operator to ϕ, unless the matrical array defining ϕ represents its effect upon a normal unit system.

§ 4. *Properties of a Real Symmetric Matrix.*

Let g be an m-tuple latent root of the real symmetric matrix ϕ, and suppose that the nullity of $\phi-g$ is less than its vacuity (which is equal to m the multiplicity of g), thus let the nullity of $\phi-g$ be $m'<m$. There will then be just m' linearly independent vectors annulled by $\phi-g$ (*i.e.*, axial to ϕ), and at least one vector annulled by $(\phi-g)^2$, but not by $\phi-g$;[*] let the former be denoted by $\rho_1, \rho_2, \ldots \rho_{m'}$, and the latter by σ. Then

$$(\phi-g)\,\rho_1 = (\phi-g)\,\rho_2 = \ldots = (\phi-g)\,\rho_{m'} = 0,$$

$$(\phi-g)\,\sigma = t_1\rho_1+t_2\rho_2+\ldots+t_{m'}\rho_{m'},$$

the t's not being all zero. Since any expression linear in the ρ's is also axial to ϕ, these may obviously be so taken that

$$(\phi-g)\,\sigma = \rho_1,$$

in which case

$$S\rho_1^2 = S\rho_1(\phi-g)\,\sigma = S\sigma(\check{\phi}-g)\,\rho_1\dagger = S\sigma(\phi-g)\,\rho_1 = 0,$$

and consequently ρ_1 is an imaginary vector. If

$$\rho_1 = \xi+\eta\sqrt{-1},$$

where ξ and η are real, since ϕ and g are real,

$$(\phi-g)(\xi+\eta\sqrt{-1}) = (\phi-g)\,\rho_1 = 0$$

gives $$\phi\xi = g\xi, \quad \phi\eta = g\eta,$$

therefore $$(\phi-g)(\xi-\eta\sqrt{-1}) = 0;$$

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[*] A. Buchheim, *Proc. Lond. Math. Soc.*, xvi., p. 75; also H. Taber, *Amer. Jour. of Math.*, Vol. xii., p. 363.

† The transverse of the sum of two matrices is equal to the sum of their transverses; a scalar is, of course, a self-transverse matrix; therefore

$$\overset{\smile}{(\phi-g)} = \check{\phi}-g.$$

therefore
$$\begin{aligned} S(\xi^2+\eta^2) &= S(\xi-\eta\sqrt{-1})(\xi+\eta\sqrt{-1}) \\ &= S(\xi-\eta\sqrt{-1})(\phi-g)\sigma \\ &= S\sigma(\phi'-g)(\xi-\eta\sqrt{-1}) \\ &= S\sigma(\phi-g)(\xi-\eta\sqrt{-1}) \\ &= 0, \end{aligned}$$
which is impossible, since then $\xi=0$, $\eta=0$, and consequently $\rho_1=0$. Therefore m' cannot be less than m, and so the vacuity of $\phi-g$ cannot exceed its nullity: *i.e.*, if *e.g.* vacuity $=2$, then nullity $=2$; if vacuity $=3$, then nullity $=3$, or, what is the same thing, the nullity is equal to the vacuity.*

It follows from what has just been proved that, if the distinct latent roots of ϕ are g_1, an m-tuple latent root, g_2 an n-tuple latent root, &c., finally g_r a p-tuple latent root, then there are m linearly independent vectors $\rho_1, \rho_2, \dots \rho_m$, which satisfy the equation
$$(\phi-g_1)\rho=0,$$

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* Prof. Cayley remarks that a "simple instance [of the theorem] is, if the real symmetric matrix
$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$
has two latent roots each $=0$, and therefore a vacuity $=2$, then it has also a nullity $=2$ [which may be shown as follows], viz., the conditions for a vacuity $=2$ are
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0, \quad bc+ca+ab-f^2-g^2-h^2=0,$$
or, if as usual the determinant is called K, and
$$(A, B, C, F, G, H) = (bc-f^2, ca-g^2, \dots),$$
then, if
$$K=0, \quad A+B+C=0,$$
these equations give
$$BC-F^2=Ka=0, \quad CA-G^2=Kb=0, \quad AB-H^2=Kc=0,$$
i.e.,
$$BC=F^2, \quad CA=G^2, \quad AB=H^2;$$
and therefore
$$\begin{aligned} A(A+B+C) &= A^2+H^2+G^2, \\ B(A+B+C) &= H^2+B^2+F^2, \\ C(A+B+C) &= G^2+F^2+C^2, \end{aligned}$$
or if $A+B+C=0$, then for real values
$$A=B=C=F=G=H=0,$$
i.e., nullity $=2$."

and may be taken all real; in addition, n real linearly independent vectors $\sigma_1, \sigma_2, \dots \sigma_n$, which are annulled by $\phi - g_2$, &c.; finally, p real linearly independent vectors $\tau_1, \tau_2, \dots \tau_p$ which are annulled by $\phi - g_i$. Since any expression $\Sigma t_r \rho_r$ linear in the ρ's is also annulled by $\phi - g_1$, it is obvious that the ρ's may be so chosen as to constitute a real normal system of unit vectors, and likewise with respect to the σ's, &c. In regard to any two vectors axial to ϕ, and corresponding to different latent roots, it is easily shown that they are normal; thus

$$O = S\rho_1 (\phi - g_2)\, \sigma_1 = S\sigma_1 (\phi - g_2)\, \rho_1 = (g_2 - g_1)\, S\rho_1 \sigma_1,$$

i.e.,

$$S\rho_1 \sigma_1 = 0.^*$$

Therefore the system of vectors comprising the ρ's, the σ's, &c. may be regarded as constituting a real normal system of unit vectors; and, as this system consists of

$$m + n + \dots + p = \omega \text{ vectors},$$

any vector in the ω-way space may be expressed linearly in terms of them.

Let the ω vectors

$$\iota_1 = (1, 0, \dots 0), \quad \iota_2 = (0, 1, \dots 0), \quad \dots \; \iota_\omega = (0, 0, \dots 1)$$

constitute a normal system of unit vectors; if ϕ is, as before, the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega} \end{pmatrix},$$

then

$$\phi\iota_1 = (a_{11}, a_{21}, \dots a_{\omega 1}) = a_{11}\iota_1 + a_{21}\iota_2 + \dots + a_{\omega 1}\iota_\omega,$$

$$\phi\iota_2 = (a_{12}, a_{22}, \dots a_{\omega 2}) = a_{12}\iota_1 + a_{22}\iota_2 + \dots + a_{\omega 2}\iota_\omega,$$

&c.

Whence, if

$$\rho = x_1\iota_1 + x_2\iota_2 + \dots + x_\omega\iota_\omega = (x_1, x_2, \dots x_\omega),$$

$$\phi\rho = \Sigma_r a_{1r}x_r \,.\, \iota_1 + \Sigma_r a_{2r}x_r \,.\, \iota_2 + \dots \Sigma_r a_{\omega r}x_r \,.\, \iota_\omega$$

$$= (\Sigma_r a_{1r}x_r, \; \Sigma_r a_{2r}x_r, \; \dots \; \Sigma_r a_{\omega r}x_r).$$

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* Compare Tait, *Elements of Quaternions*, Chap. v., Art. 363.

Let $$q = \omega - p,$$

and suppose that

$$\rho_1 = \varpi\iota_1, \quad \rho_2 = \varpi\iota_2, \ \ldots \ \rho_m = \varpi\iota_m,$$

or, expressed in a more compact form,

$$(\rho_1, \rho_2, \ldots \rho_m) = (\varpi \between \iota_1, \iota_2, \ldots \iota_m);$$

suppose also that

$$(\sigma_1, \sigma_2, \ldots \sigma_n) = (\varpi \between \iota_{m+1}, \iota_{m+2}, \ldots \iota_{m+n}),$$

&c.,

$$(\tau_1, \tau_2, \ldots \tau_n) = (\varpi \between \iota_{q+1}, \iota_{q+2}, \ldots \iota_{q+p}).$$

Let $[r, s]$ denote the matrix whose constituent in the r^{th} ro s^{th} column is unity, and of which all the other constituent zero; and let

$$\theta = \sum_1^m {}_r\, g_1 [r, r] + \sum_{m+1}^{m+n} {}_r\, g_2 [r, r] + \ldots + \sum_{q+1}^{q+p} {}_r\, g_i [r, r].$$

Moreover, let it be supposed that

$$\begin{aligned} &(\theta \between \rho_1, \rho_2, \ldots \rho_m, \sigma_1, \sigma_2, \ldots \sigma_n, \ldots \tau_1, \tau_2, \ldots \tau_p) \\ &= (g_1\rho_1, g_1\rho_2, \ldots g_1\rho_m, g_2\sigma_1, g_2\sigma_2, \ldots g_2\sigma_n, \ldots g_i\tau_1, g_i\tau_2, \ldots g_i\tau \\ &= (\phi \between \rho_1, \rho_2, \ldots \rho_m, \sigma_1, \sigma_2, \ldots \sigma_n, \ldots \tau_1, \tau_2, \ldots \tau_p); \end{aligned}$$

it is then easy to see that

$$\begin{aligned} (\theta \between \varpi \between \iota_1, \iota_2, \ldots \iota_m) &= (\theta \between \rho_1, \rho_2, \ldots \rho_m) \\ &= (\phi \between \rho_1, \rho_2, \ldots \rho_m) \\ &= (\phi \between \varpi \between \iota_1, \iota_2, \ldots \iota_m), \\ (\theta \between \varpi \between \iota_{m+1}, \iota_{m+2}, \ldots \iota_{m+n}) &= (\theta \between \sigma_1, \sigma_2, \ldots \sigma_n) \\ &= (\phi \between \sigma_1, \sigma_2, \ldots \sigma_n) \\ &= (\phi \between \varpi \between \iota_{m+1}, \iota_{m+2}, \ldots \iota_{m+n}), \\ \text{\&c.} \qquad & \qquad \text{\&c.,} \\ (\theta \between \varpi \between \iota_{q+1}, \iota_{q+2}, \ldots \iota_\omega) &= (\theta \between \tau_1, \tau_2, \ldots \tau_p) \\ &= (\phi \between \tau_1, \tau_2, \ldots \tau_p) \\ &= (\phi \between \varpi \between \iota_{q+1}, \iota_{q+2}, \ldots \iota_\omega), \end{aligned}$$

i.e., $$(\text{[illegible]}\ \iota_\omega) = (\text{[illegible]}\ \iota_2, \ldots \iota_\omega).$$

Therefore $$\theta\varpi = \phi\varpi;$$

but, since the ρ's, σ's, &c. are linearly independent, the determinant of ϖ is not null; therefore

$$\phi = \varpi^{-1}\theta\varpi.$$

Moreover, if $\rho = \Sigma_r x_r \iota_r$ and $\sigma = \Sigma_r y_r \iota_r$,

$$\begin{aligned} S\rho\sigma &= S(\Sigma_r x_r \iota_r)(\Sigma_r y_r \iota_r) \\ &= \Sigma_r x_r y_r \\ &= S(\overset{m}{\underset{1}{\Sigma}}{}_r x_r \rho_r + \overset{n}{\underset{1}{\Sigma}}{}_r x_{m+r} \sigma_r + \ldots)(\overset{m}{\underset{1}{\Sigma}}{}_r y_r \rho_r + \overset{n}{\underset{1}{\Sigma}}{}_r y_{m+r} \sigma_r + \ldots) \\ &= S(\Sigma_r x_r \varpi\iota_r)(\Sigma_r y_r \varpi\iota_r) \\ &= S\varpi\rho\varpi\sigma \\ &= S\rho\breve{\varpi}\varpi\sigma \end{aligned}$$

for all values of ρ, therefore $\sigma = \breve{\varpi}\varpi\sigma$

for all values of σ, *i.e.*, $\breve{\varpi}\varpi = 1$.

Consequently, for any symmetric matrix ϕ it is always possible to find a real orthogonal matrix ϖ, such that

$$\phi = \breve{\varpi}\theta\varpi.$$

If $$\eta = \overset{m}{\underset{1}{\Sigma}}{}_r g_1^{1/\mu}[r, r] + \overset{m+n}{\underset{m}{\Sigma}}{}_r g_2^{1/\mu}[r, r] + \ldots + \overset{q+p}{\underset{q}{\Sigma}}{}_r g_i^{1/\mu}[r, r],$$

and if $$\psi = \breve{\varpi}\,\eta\,\varpi,$$

then ψ is a μ^{th} root of ϕ, *i.e.*, $\psi^\mu = \phi$.

Therefore a symmetric matrix has always a μ_{th} root that is also a symmetric matrix. When two or more latent roots of ϕ are equal, there exist other μ^{th} roots of ϕ, which are not representable as above; these μ^{th} roots are not, in general, symmetric matrices. Of the μ^{th} roots of ϕ representable as above, certain are given by Sylvester's formula (*Johns Hopkins Univ. Circs.*, Vol. III., p. 34), viz., those for which for every latent root the same value of its μ^{th} root is taken throughout in the summation. With respect to any function of ϕ representable by Sylvester's formula, it is, of course, also a symmetric matrix. If the latent roots of ϕ are all distinct, the only μ^{th} roots of ϕ are those given by Sylvester's formula, and therefore they are all symmetric matrices.

§ 5. *Properties of a Real Skew Symmetric Matrix.*

Let ϕ be a real skew symmetric matrix of which $g\sqrt{-1}$ is an m-tuple latent root; suppose that the nullity of $\phi - g\sqrt{-1}$ is less than its vacuity m, thus let the nullity of $\phi - g\sqrt{-1}$ be $m' < m$. There will then be just m' linearly independent vectors, $\rho_1, \rho_2, \dots \rho_{m'}$, annulled by $\phi - g\sqrt{-1}$, and at least one vector σ such that the linear vector function $\phi\sigma$ is expressible linearly in terms of the ρ's;* and these may be so chosen that

$$(\phi - g\sqrt{-1})\,\sigma = \rho_1.$$

If $g \neq 0$, since ϕ and g are real, each of the axial vectors ρ is imaginary; if $g = 0$, then

$$S\rho_1^2 = S\rho_1\phi\sigma = S\sigma\check{\phi}\rho_1 = -S\sigma\phi\rho_1 = 0;$$

consequently, in either case ρ_1 is imaginary.

Let $$\rho_1 = \xi + \eta\sqrt{-1},$$

where ξ and η are real; if $g \neq 0$, then $\xi - \eta\sqrt{-1}$ is an axial vector to ϕ corresponding to the latent root $-g\sqrt{-1}$; if $g = 0$, then, since

$$\phi\,(\xi + \eta\sqrt{-1}) = \phi\rho_1 = 0,$$

therefore $$\phi\xi = 0, \quad \phi\eta = 0;$$

hence it follows that $$\phi\,(\xi - \eta\sqrt{-1}) = 0.$$

Therefore
$$\begin{aligned} S\,(\xi^2 + \eta^2) &= S\,(\xi - \eta\sqrt{-1})(\xi + \eta\sqrt{-1}) \\ &= S\,.\,(\xi - \eta\sqrt{-1})(\phi - g\sqrt{-1})\,\sigma \\ &= S.\sigma\,(\check{\phi} - g\sqrt{-1})(\xi - \eta\sqrt{-1}) \\ &= -S.\sigma\,(\phi + g\sqrt{-1})(\xi - \eta\sqrt{-1}) \\ &= 0, \end{aligned}$$

which is impossible, since then $\xi = 0$, $\eta = 0$, and consequently $\rho_1 = 0$. Whence it follows that $m' = m$, and therefore for any latent root $g\sqrt{-1}$ of ϕ, the nullity of $\phi - g\sqrt{-1}$ is equal to its vacuity.

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* See footnote *, p. 457.

The axial vectors of a real skew symmetric matrix that correspond to a non-zero latent root all satisfy the equation

$$S\rho^2 = 0,$$

and so may be termed *circular rays*, or rays to the absolute, *i.e.*, vectors from the origin to the imaginary $(\omega-1)$-way spherical spread at infinity. Thus, for the axial vector

$$\rho_1 = \xi+\eta\sqrt{-1},$$

corresponding to the non-zero latent root $g\sqrt{-1}$,

$$\begin{aligned}-2g\sqrt{-1}\,S\,(\xi+\eta\sqrt{-1})^2 &= -S\,(\xi+\eta\sqrt{-1})(\varphi+g\sqrt{-1})(\xi+\eta\sqrt{-1})\\ &= -S\,(\xi+\eta\sqrt{-1})(\check{\phi}+g\sqrt{-1})(\xi+\eta\sqrt{-1})\\ &= \quad S\,(\xi+\eta\sqrt{-1})(\phi-g\sqrt{-1})(\xi+\eta\sqrt{-1})\\ &= 0.\end{aligned}$$

If ρ_1 is axial to ϕ and corresponds to the latent root $g_1\sqrt{-1}$, and if ρ_2 is axial to ϕ and corresponds to $g_2\sqrt{-1}$, then

$$\begin{aligned}(g_2+g_1)\sqrt{-1}\,.\,S\rho_1\rho_2 = S\rho_1\,(\phi+g_1\sqrt{-1})\,\rho_2 &= S\rho_2\,(\check{\phi}+g_1\sqrt{-1})\,\rho_1\\ &= -S\rho_2\,(\phi-g_1\sqrt{-1})\,\rho_1 = 0.\end{aligned}$$

Therefore an axial vector of a real skew symmetric matrix corresponding to one latent root is normal to the axial vector corresponding to any other latent root not the conjugate imaginary of the first.

By what has been shown above, it follows that, if the distinct latent roots of ϕ are $g_1\sqrt{-1}$, an m-tuple latent root, $g_2\sqrt{-1}$ an n-tuple latent root, &c., then there are m linearly independent vectors $\rho_1, \rho_2, \dots \rho_m$, annulled by $\phi-g_1\sqrt{-1}$, n linearly independent vectors $\sigma_1, \sigma_2, \dots \sigma_n$, annulled by $\phi-g_2\sqrt{-1}$, &c. It cannot happen that the ρ's, the σ's, &c. are linearly related; for suppose

$$(x_1\rho_1+x_2\rho_2+\dots+x_m\rho_m)+(y_1\sigma_1+y_2\sigma_2+\dots+y_n\sigma_n)+\&c. = 0,$$

operating upon the left-hand member of this equation by the product of ϕ less each of its latent roots except $g_1\sqrt{-1}$, all the terms in this sum of vector will be annulled except the first m, and these will all be multiplied by the product of $g_1\sqrt{-1}$ less each of the other latent roots of ϕ; consequently,

$$x_1\rho_1+x_2\rho_2+\dots+x_m\rho_m = 0,$$

therefore $$x_1 = x_2 = \ldots = x_m = 0.$$

In the same way it may be shown that the y's are all zero, &c. Following a method similar to that of § 4, it may now be shown that, if

$$\theta = g_1\sqrt{-1}\,\overset{m}{\underset{1}{\Sigma}}{}_r\,[r, r] + g_2\sqrt{-1}\,\overset{m+n}{\underset{m+1}{\Sigma}}{}_r\,[r, r] + \&c.,$$

a matrix ϖ can always be found such that

$$\phi = \varpi^{-1}\,\theta\varpi.$$

Whence, as before, it follows that a real skew symmetric matrix has always a μ^{th} root. Thus, if

$$\eta = \lambda_1 g_1^{1/\mu}\,\overset{m}{\underset{1}{\Sigma}}{}_r\,[r, r] + \lambda_2 g_2^{1/\mu}\,\overset{m+n}{\underset{m+1}{\Sigma}}{}_r\,[r, r] + \&c.,$$

where $$\lambda_1^\mu = \lambda_2^\mu = \ldots = \sqrt{-1},$$

and if $$\psi = \varpi^{-1}\eta\varpi,$$

then $$\psi^\mu = \phi.$$

Of the μ^{th} roots of ϕ representable as above certain are given by Sylvester's formula.

§ 6. *Nullity of $\phi - g$, when $\check{\phi}\phi = 1$, and g is a multiple latent root of ϕ.*

If ϕ is a real orthogonal matrix of which -1 is not a latent root, we may put

$$\Upsilon = \frac{\phi - 1}{\phi + 1}.$$

Therefore

$$\check{\Upsilon} = \frac{1}{\check{\phi}+1}(\check{\phi}-1) = \frac{1}{\phi^{-1}+1}(\phi^{-1}-1) = \frac{1}{1+\phi}(1-\phi) = \frac{1-\phi}{1+\phi} = -\Upsilon.$$

The first equation gives

$$\phi = (1-\Upsilon)^{-1}(1+\Upsilon).^*$$

Suppose that g is an m-tuple latent root of ϕ, but that the nullity of ϕ is $m' < m$. There will then be m' linearly independent vectors

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* Cayley, *Phil. Trans.*, 1858.

$\rho_1, \rho_2, \dots \rho_m$, annulled by $\phi - g$, and at least one vector σ, for which, if the ρ's are properly chosen,

$$(\phi - g)\,\sigma = \rho_1.$$

Whence it follows that

$$(1-\Upsilon)^{-1}(1+\Upsilon)\,\rho_1 = \phi\rho_1 = g\rho_1,$$

i.e.,

$$(1+\Upsilon)\,\rho_1 = g\,(1-\Upsilon)\,\rho_1,$$

therefore

$$\Upsilon\rho_1 = \frac{g-1}{g+1}\,\rho_1;$$

likewise

$$(1-\Upsilon)^{-1}(1+\Upsilon)\,\sigma = \phi\sigma = g\sigma + \rho_1,$$

therefore

$$\Upsilon\sigma = \frac{g-1}{g+1}\,\sigma + \frac{2}{(g+1)^2}\,\rho_1.$$

The vector ρ_1 is axial to the real skew symmetric matrix Υ, and consequently is imaginary, unless

$$\frac{g-1}{g+1} = 0,$$

which is contrary to supposition. Let

$$\rho_1 = \xi + \eta\sqrt{-1},$$

ξ and η being real; corresponding to the latent root

$$\frac{1-g}{1+g} = -\frac{g-1}{g+1}$$

$\left(\text{the conjugate imaginary to } \dfrac{g-1}{g+1}\right)$ is then the axial vector to Υ conjugate to ρ_1, viz., $\xi - \eta\sqrt{-1}$. Whence

$$\begin{aligned}\frac{2}{(g+1)^2}\,S\,(\xi^2+\eta^2) &= \frac{2}{(g+1)^2}\,S\,(\xi-\eta\sqrt{-1})(\xi+\eta\sqrt{-1}) \\ &= S\,(\xi-\eta\sqrt{-1})\left(\Upsilon - \frac{g-1}{g+1}\right)\sigma \\ &= S\,\sigma\left(\check{\Upsilon} - \frac{g-1}{g+1}\right)(\xi-\eta\sqrt{-1}) \\ &= -S\,\sigma\left(\Upsilon - \frac{1-g}{1+g}\right)(\xi-\eta\sqrt{-1}) \\ &= 0,\end{aligned}$$

which is impossible, since then $\xi = 0$, $\eta = 0$, and so $\rho_1 = 0$, and therefore $m' = m$.

If the distinct latent roots of ϕ are g_1, an m-tuple latent root, g_2 an n-tuple latent root, &c., and if

$$\theta = g_1 \overset{m}{\underset{1}{\Sigma_r}} [r, r] + g_2 \overset{m+n}{\underset{m+1}{\Sigma_r}} [r, r] + \&\text{c.},$$

it follows, from the theorem just proved, that a matrix ϖ can always be found such that

$$\phi = \varpi^{-1} \theta \varpi.$$

Whence, if $$\eta = g_1^{1/\mu} \overset{m}{\underset{1}{\Sigma_r}} [r, r] + g_2^{1/\mu} \overset{m+n}{\underset{m+1}{\Sigma_r}} [r, r] + \&\text{c.},$$

and if $$\psi = \varpi^{-1} \eta \varpi.$$

therefore $$\psi^\mu = \phi.$$

If -1 is a latent root of ϕ, but none of the latent roots of ϕ are equal to unity, we may put

$$\Upsilon = \frac{\phi + 1}{\phi - 1},$$

and proceed as before.

It is very readily shown, as in the last section, that all axial vectors except those corresponding to the latent roots ± 1 are circular rays, and that an axial vector corresponding to any one latent root is normal to the axial vector corresponding to any other latent root not the reciprocal of the first. Thus, if ρ_1 and ρ_2 are axial vectors to ϕ, corresponding respectively to g_1 and g_2, then

$$\left(g_1 - \frac{1}{g_1}\right) S\rho_1^2 = S\rho_1 \left(\phi - \frac{1}{g_1}\right) \rho_1 = S.\rho_1 \left(\breve{\phi} - \frac{1}{g_1}\right) \rho_1$$

$$= S.\rho_1 \phi^{-1} \left(1 - \frac{\phi}{g_1}\right) \rho_1 = 0,$$

and $$\left(g_2 - \frac{1}{g_1}\right) S\rho_1\rho_2 = S\rho_1 \left(\phi - \frac{1}{g_1}\right) \rho_2 = S.\rho_2 \left(\breve{\phi} - \frac{1}{g_1}\right) \rho_1$$

$$= S.\rho_2 \phi^{-1} \left(1 - \frac{\phi}{g_1}\right) \rho_1 = 0$$

[*Postscript. December 4th*, 1891.

In § 6, it was shown that, if ϕ is a real orthogonal matrix which has not both ± 1 as latent roots, then for a multiple latent root g of ϕ, the nullity of $\phi - g$ is equal to its vacuity. This theorem, however, is true for any orthogonal matrix. A demonstration will be given below. This depends upon the lemma, which I shall first prove, that the modulus of every latent root of a real orthogonal matrix ϕ is equal to unity. Thus, if g is a latent root of ϕ, then

$$\text{Det. } (\phi-g) = 0,$$

and consequently, by the law of multiplication of matrices,

$$\begin{aligned} \text{Det. } \left[1+g(\phi-\check{\phi})-g^2\right] &= \text{Det. } \left[(\phi-g)(\check{\phi}+g)\right] \\ &= \text{Det. } (\phi-g) \,.\, \text{Det. } (\check{\phi}+g) \\ &= 0, \end{aligned}$$

i.e.,
$$\text{Det. } \left[(\phi-\check{\phi})-\frac{g^2-1}{g}\right] = 0;$$

but $\phi-\check{\phi}$ is a real skew symmetric matrix, therefore its latent root $\frac{g^2-1}{g}$ is either zero or a pure imaginary, *i.e.*, g, if real, $= \pm 1$. We have, moreover,

$$\begin{aligned} \text{Det. } \left[1-g(\phi+\check{\phi})+g^2\right] &= \text{Det. } \left[(\phi-g)(\check{\phi}-g)\right] \\ &= \text{Det. } (\phi-g) \,.\, \text{Det. } (\check{\phi}-g) \\ &= 0, \end{aligned}$$

i.e.,
$$\text{Det. } \left[(\phi+\check{\phi})-\frac{g^2+1}{g}\right] = 0;$$

but $\phi+\check{\phi}$ is a real symmetric matrix; therefore its latent root $\frac{g^2+1}{g}$ is real. Putting $g = a+b\sqrt{-1}$, and $\frac{g^2+1}{g} = c$, where a, b, c are real, we obtain, from the last equation,

$$a^2-b^2+1 = ac,$$

$$2ab = bc;$$

therefore, if $b \neq 0$ (*i.e.*, if g is not real), $a^2+b^2 = 1$. Or, in other words, from the first condition $\frac{g^2-1}{g} = c'\sqrt{-1}$, where c' is real, we derive the equations

$$a^2-b^2-1 = -c'b,$$

$$2ab = c'a,$$

i.e., the modulus of g is unity, or $a = 0$; but from the condition

$\frac{g^2+1}{g} = c$, as was shown, it follows that the modulus of g is unity or $b = 0$; it cannot happen that both a and b are zero, as ϕ is orthogonal, therefore the modulus of ϕ is unity.[*]

Let now g be an m-tuple latent root of the real orthogonal matrix ϕ, and suppose that the nullity of $\phi - g$ is less than its vacuity; thus, let the nullity of $\phi - g$ be $m' < m$. There will then be just m' linearly independent vectors $\rho_1, \rho_2, \ldots \rho_{m'}$, annulled by $\phi - g$, and at least one vector σ, for which, if the ρ's are properly chosen,

$$(\phi - g)\,\sigma = \rho_1.$$

If g is not equal to ± 1, g is imaginary, and then, since ϕ is real, the axial vectors ρ are imaginary; if $g = \pm 1$, then

$$\begin{aligned} S\rho_1^2 &= S\rho_1\,(\phi \mp 1)\,\sigma \\ &= S\sigma\,(\breve{\phi} \mp 1)\,\rho_1 \\ &= S\sigma\,(\phi^{-1} \mp 1)\,\rho_1 \\ &= S\sigma\,(\pm 1 \mp 1)\,\rho_1 \\ &= 0; \end{aligned}$$

consequently in either case ρ_1 is imaginary.

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[*] The following demonstration of this theorem follows more closely the general method employed in this paper. If ρ is an axial vector to ϕ, corresponding to the latent root g, then (since ϕ is orthogonal and is the transverse of $\breve{\phi}$)

$$S\rho^2 = S\,.\,\rho\breve{\phi}\phi\rho = S\,.\,\phi\rho\phi\rho = g^2 S\rho^2;$$

therefore either $g = \pm 1$ or ρ is imaginary. Let $\rho = \xi + \eta\sqrt{-1}$, ξ and η being real; if g is imaginary, let r be its modulus and θ its argument, then, corresponding to $re^{-\theta}$, the conjugate imaginary of g is the axial vector $\xi - \eta\sqrt{-1}$; if g is real, then, since ϕ is real, $\phi\xi = g\xi$, $\phi\eta = g\eta$; consequently, in either case, $\xi - \eta\sqrt{-1}$ is axial to ϕ and corresponds to the conjugate imaginary to g. Whence it follows that

$$\begin{aligned} S(\xi^2 + \eta^2) &= S(\xi + \eta\sqrt{-1})(\xi - \eta\sqrt{-1}) \\ &= S\,.\,(\xi + \eta\sqrt{-1})\,\breve{\phi}\phi\,(\xi - \eta\sqrt{-1}) \\ &= S\,.\,\phi\,(\xi + \eta\sqrt{-1})\,\phi\,(\xi - \eta\sqrt{-1}) \\ &= r^2 S\,(\xi + \eta\sqrt{-1})(\xi - \eta\sqrt{-1}) \\ &= r^2 S\,(\xi^2 + \eta^2); \end{aligned}$$

but $S(\xi^2 + \eta^2)$ cannot be zero unless $\xi = 0$, $\eta = 0$, and consequently $\rho = 0$, which is impossible; therefore $r^2 = 1$.

Let $$\rho_1 = \xi + \eta\sqrt{-1},$$

where ξ and η are real. If g is imaginary and equal to $e^{\theta\sqrt{-1}}$, then corresponding to its conjugate imaginary $e^{-\theta\sqrt{-1}}$ is the axial vector conjugate to ρ_1, viz., $\xi - \eta\sqrt{-1}$; and evidently we have

$$\begin{aligned}\phi^{-1}(\xi-\eta\sqrt{-1}) &= \frac{1}{e^{-\theta\sqrt{-1}}}(\xi-\eta\sqrt{-1}) \\ &= g\,(\xi-\eta\sqrt{-1}).\end{aligned}$$

Whence
$$\begin{aligned}S\,(\xi^2+\eta^2) &= S\,(\xi-\eta\sqrt{-1})(\xi+\eta\sqrt{-1}) \\ &= S\,.\,(\xi-\eta\sqrt{-1})(\phi-g)\,\sigma \\ &= S\,.\,\sigma\,(\check{\phi}-g)(\xi-\eta\sqrt{-1}) \\ &= S\,.\,\sigma\,(\phi^{-1}-g)(\xi-\eta\sqrt{-1}) \\ &= 0.\end{aligned}$$

Likewise, if $g = \pm 1$, then, since ϕ is real,

$$\phi\xi = \pm\xi, \quad \phi\eta = \pm\eta;$$

consequently
$$\begin{aligned}\phi\,(\xi-\eta\sqrt{-1}) &= \pm(\xi-\eta\sqrt{-1}), \\ \phi^{-1}(\xi-\eta\sqrt{-1}) &= \pm(\xi-\eta\sqrt{-1}).\end{aligned}$$

Whence
$$\begin{aligned}S\,(\xi^2+\eta^2) &= S\,(\xi-\eta\sqrt{-1})(\xi+\eta\sqrt{-1}) \\ &= S.(\xi-\eta\sqrt{-1})(\phi\mp 1)\,\sigma \\ &= S\,.\,\sigma\,(\check{\phi}\mp 1)(\xi-\eta\sqrt{-1}) \\ &= S\,.\,\sigma\,(\phi^{-1}\mp 1)(\xi-\eta\sqrt{-1}) \\ &= 0.\end{aligned}$$

But if $S\,(\xi^2+\eta^2) = 0$, then $\xi = 0$, $\eta = 0$, and consequently $\rho_1 = 0$, which is impossible. Therefore $m' = m$, *i.e.*, the nullity of $\phi - g$ i equal to its vacuity.]

A Property of the Circumcircle. By R. Tucker, M.A.

[*Read June 11th*, 1891.]

Using the notation of my paper, "The *Triplicate-Ratio* Circle" (*Quar. Jour.*, Vol. xix., No. 76), viz.,

$$k \equiv a^2+b^2+c^2, \quad \lambda^2 \equiv a^2b^2+b^2c^2+c^2a^2,$$

we have (*l.c.*, p. 347)

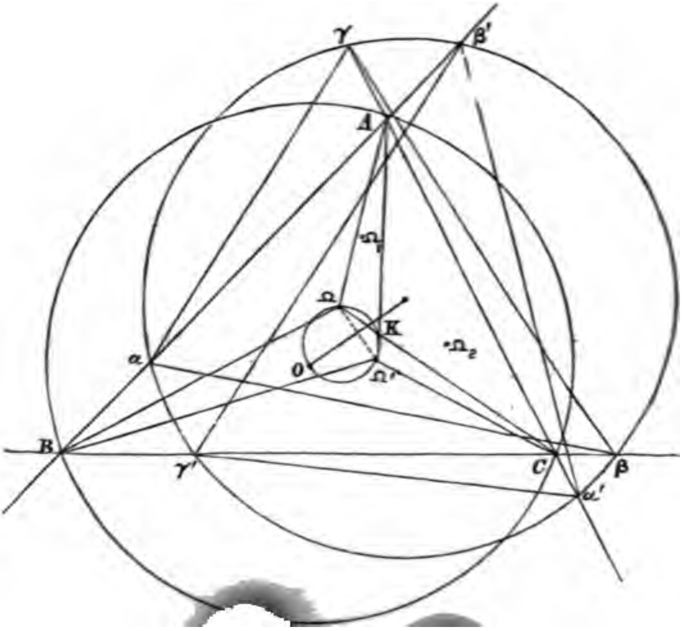
$$A\Omega = cb^2/\lambda, \quad B\Omega = ac^2/\lambda, \quad C\Omega = ba^2/\lambda\,;$$

$$A\Omega' = bc^2/\lambda, \quad B\Omega' = ca^2/\lambda, \quad C\Omega' = ab^2/\lambda.$$

In the figure, I take

$$\Omega\alpha A = \Omega\beta B = \Omega\gamma c = \omega,$$

and $$\Omega'\alpha' A = \Omega'\beta' B = \Omega'\gamma' C = \omega.$$



Then, using trilinear coordinates, we have

$$\begin{array}{lllll} \text{point } (\alpha) & \omega_b, & abk, & 0 \\ \quad ,, \;\; (\beta) & 0, & \omega_c, & bck \\ \quad ,, \;\; (\gamma) & cak, & 0, & \omega_a \end{array} \left.\vphantom{\begin{array}{l}a\\b\\c\end{array}}\right\} \text{mod.} \begin{array}{l} a\lambda^2, \\ b\lambda^2, \\ c\lambda^2, \end{array}$$

$$\begin{array}{lllll} \quad ,, \;\; (\alpha') & \omega_c, & 0, & cak \\ \quad ,, \;\; (\beta') & abk, & \omega_a, & 0 \\ \quad ,, \;\; (\gamma') & 0, & bck, & \omega_b \end{array} \left.\vphantom{\begin{array}{l}a\\b\\c\end{array}}\right\} \text{mod.} \begin{array}{l} a\lambda^2, \\ b\lambda^2, \\ c\lambda^2, \end{array}$$

where
$$\left.\begin{array}{l} \omega_a \equiv b^2c^2-a^4, \\ \omega_b \equiv c^2a^2-b^4, \\ \omega_c \equiv a^2b^2-c^4. \end{array}\right\}$$

Hence $$B\beta = ac^2k/\lambda^2, \quad B\alpha = c\omega_b/\lambda^2;$$

therefore $$(\alpha\beta)^2 = c^2\left[a^2c^2k^2+\omega_b^2-k\omega_b(k-2b^2)\right]/\lambda^4 = c^2,$$

and $$\alpha\beta = c.$$

In like manner, we can show
$$\alpha'\beta' = c;$$
hence the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are congruent with the triangle ABC.

If the equation to the circle $\alpha\beta\gamma$ be
$$\alpha\beta\gamma+\ldots+\ldots = (a\alpha+\ldots+\ldots)(p\alpha+q\beta+r\gamma),$$
we have $$abck\omega_b = a\lambda^2(p\omega_b+qabk), \text{ \&c.},$$
whence $$p = kbc\omega_a/\lambda^4,$$
i.e., the circle is
$$\alpha\beta\gamma+\ldots+\ldots = (a\alpha+\ldots+\ldots)\,k\left[bc\omega_a\alpha+\ldots+\ldots\right]/\lambda^4.$$

It is readily proved that α', β', γ', are on this circle; hence *the common circumcircle of* $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ *is equal to the circumcircle of* ABC.

The radical axis of the two circles is
$$bc\omega_a\alpha+\ldots+\ldots = 0,$$
i.e., the line through the Brocard points.

The centre of the circle is then the image of the circumcentre (O) of ABC in $\Omega\Omega'$.

Since* $\Omega\alpha = \Omega A$, $\Omega\beta = \Omega B$; therefore the triangles $\Omega\alpha\beta$, ΩAB are congruent, and therefore

$$\angle\, \Omega\alpha\beta = \omega;$$

hence Ω is the positive Brocard point of $\alpha\beta\gamma$, and

$$\angle\, A\alpha\beta = 2\omega;$$

hence
$$\beta'\alpha\gamma = A - 2\omega\ (= \Omega A \Omega').$$

If now Ω_1 be the negative Brocard point of $\alpha\beta\gamma$, we have

$$\alpha\Omega_1 = A\Omega' = bc^2/\lambda, \quad \Omega_1 \alpha A = 3\omega - A;$$

hence the co-ordinates of Ω_1 are

$$a^2c \sin(3\omega - B)/\lambda, \quad b^2a \sin(3\omega - C)/\lambda, \quad c^2b \sin(3\omega - A)/\lambda.$$

Similarly Ω', Ω_2 are the negative and positive Brocard points of $\alpha'\beta'\gamma'$; hence Ω_2 is given by

$$a^2b \sin(3\omega - C)/\lambda, \quad b^2c \sin(3\omega - A)/\lambda, \quad c^2a \sin(3\omega - B)/\lambda.$$

$\Omega\Omega_2$, $\Omega'\Omega_1$ intersect in K, and $\Omega_1\Omega_2$ is parallel to $\Omega\Omega'$, and other results follow from the symmetry of the figure.

We have $\alpha'\beta$ parallel to AB, &c.; and mid-point of $\alpha'\beta$ lies on median of ABC through C, &c.

If the intersection of $B\Omega_1$, $C\Omega_2$ be joined to A, then this and the like lines from B and C meet in

$$\alpha/\sin(3\omega - A) = \beta/\sin(3\omega - B) = \gamma/\sin(3\omega - C);$$

and, if in like manner we join A (&c.) to the intersection of $B\Omega_2$, $C\Omega_1$, (&c.), these lines meet in

$$\alpha \sin^2(3\omega - A)/a^3 = \beta \sin^2(3\omega - B)/b^3 = \gamma \sin^2(3\omega - C)/c^3.$$

The following presents to the Library were received during the Recess:—

"Royal Society—Proceedings," Vol. xlix., Nos. 300, 301, 302.

"Educational Times," for July, August, September, and October.

"Education," Vol. ii., Nos. 7, 8, 9, 10.

"Journal of the Institute of Actuaries," Vol. xxix., Parts 3 and 4.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. iv., Nos. 4 and 5.

"Proceedings of the Royal Society of Edinburgh," Vol. xvii.

"Royal Irish Academy—Proceedings," Vol. i., No. 5; "Transactions," Vol. xxix., Part 15.

"Royal Dublin Society—Scientific Proceedings," Vol. vi., Part 10; Vol. vii., Parts 1 and 2; "Scientific Transactions," Vol. iv., Parts 6, 7, and 8.

"An Elementary Treatise on the Integral Calculus," by Benjamin Williamson, Sixth Edition, 8vo; London, 1891.

"Smithsonian Report," 1889, 8vo; Washington, 1890.

"Annals of Mathematics," Vol. vi., Nos. 1 and 2.

"Experiments in Aerodynamics," by S. P. Langley, 4to; Washington, 1891.

"Report of the Director and Treasurer of the Michigan Mining School, 1886–1891," 8vo; Marquette, Mich., 1891.

"Halsted's Lobatschewsky's Geometry," 8vo; Austin, 1891.

"Documents relating to the fixing of a Standard of Time and the Legalization thereof," 8vo; Ottawa, 1891.

"Bulletin des Sciences Mathématiques," Tome xv., May, June, July, 1891.

"Recherches expérimentales aérodynamiques et données d'expérience," par M. S. P. Langley, 4to pamphlet ("Comptes Rendus," July, 1891).

"Annales de la Faculté des Sciences de Toulouse," Tome iv., Fasc. 2, 3, and 4; Tome v., Fasc. 1 and 2.

"Annales de l'Ecole Polytechnique de Delft," Tome vi., Livr. 3 and 4; Tome vii., Livr. 1.

"Bulletin de la Société Mathématique de France," Tome xix., Nos. 5 and 6.

M. Emile Lemoine:—"Contributions à la Géométrie du Triangle;" "Sur les Triangles Orthologiques et sur divers sujets de la Géométrie du Triangle," 8vo pamphlets.

"Acta Mathematica," xv., 1 and 2.

"Annali di Matematica," Tomo xix., Fasc. 3.

"Rendiconti del Circolo Matematico di Palermo," Tomo v., Fasc. 4 and 5.

"Memorie della Regia Accademia di Scienze, Lettere, ed Arti in Modena," Vol. vii.

"Journal für die reine und angewandte Mathematik," Band cviii., Heft 3 and 4.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 131, 132, 134, 135, 136, 137, 138, and 2 Nos. of Index.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vii., Fasc. 9, 10, 11, 12, 1° Sem.; Fasc. 1, 2, 3, 4, 5, 2° Sem.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1891, i. to xl.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stücke 6, 7, 8.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxv., 2^{me} Livraison.

"Jahrbuch über die Fortschritte der Mathematik," Band xx., Heft 3; 1891.

"Nieuw Archief voor Wiskunde," Deel xviii., Stuk 1 and 2.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1891, I. and II.

"Sitzungsberichte der Physikalisch-Medicinischen Societät in Erlangen," 23 Heft, 1891.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," XXXVI., Heft 1, 1891.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. X., No. 2; 1891, Coimbra.

"Memorias y Revista de la Sociedad Cientifica 'Antonio Alzate,'" Tomo IV., Nos. 9 and 10, 1891.

"Abhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften," XVII., Nos. 3, 4, and 5.

"Neue Rechnungsmethoden der höheren Mathematik," von Dr. Julius Bergbohm, 8vo pamphlet; Stuttgart, 1891.

"Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden," door D. Bierens de Haan, 8vo pamphlet; Amsterdam, 1891.

"Bibliographie de l'Histoire des Sciences Mathématiques aux Pays-Bas," par D. Bierens de Haan, 8vo pamphlet; Stockholm, 1891.

"Bulletins de l'Académie Royale de Belgique," 5me Série, Tomes XVIII. (1889), XIX., XX. (1890), and XXI. (1891).

"Annuaire de l'Académie Royale de Belgique," 1890 and 1891.

"The Journal of the College of Science, Imperial University, Japan," Vol. IV., Part 1; 1891.

APPENDIX.

Many of the figures shown by Mr. Perigal (p. 172) are to be seen in "Graphic Demonstrations of Geometric Problems," selected by Henry Perigal, F.R.A.S., issued by the Association for the Improvement of Geometrical Teaching (1891). See also article by the same gentleman, "On Geometric Dissections and Transformations," in the *Messenger of Mathematics*, No. 19, 1872.

With reference to Dr. Hill's paper (p. 214) attention may be called to a paper by the same author, communicated to the Royal Society, entitled "On the Locus of Singular Points and Lines which occur in connexion with the Theory of the Locus of Ultimate Intersections of a System of Surfaces" (see Abstract, Vol. L., of *Royal Society's Proceedings*).

Professor Cayley has sent the following Note for publication:—

In my paper, "Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points," *Proc. Lond. Math. Soc.*, t. III. (1870), pp. 196–198, I notice a difficulty which presents itself in the theory. The transformation is given by the equations

$$x' : y' : z' = X : Y : Z,$$

where X, Y, Z are functions $(*\between x, y, z)^n$, such that $X=0$, $Y=0$, $Z=0$ are curves in the first plane passing through a_1 points each once, a_2 points each twice (that is, having each of the a_2 points for a double point), a_3 points each 3 times, and so on. We have as the condition of a single variable point of intersection,

$$a_1+4a_2+9a_3+\ldots = n^2-1,$$

and as the condition in order that each of the curves $X=0$, $Y=0$, $Z=0$, or say the curve $aX+bY+cZ=0$, may be unicursal,

$$a_2+3a_3+\ldots = \tfrac{1}{2}(n-1)(n-2);$$

and we thence deduce

$$a_1+3a_2+6a_3+\ldots = \tfrac{1}{2}n(n+3)-2;$$

viz., the postulation of the fixed points *quoad* a curve of the order n is less by 2 than the postulandum (or, as I prefer to call it, the capacity) $\tfrac{1}{2}n(n+3)$ of the curve of the order n; that is, there are precisely the three asyzygetic curves $X=0$, $Y=0$, $Z=0$. This is as it should be, assuming that the $(a_1, a_2, a_3 \ldots)$ points are an ordinary system of points: but what if they form a special system having a postulation less than $a_1+3a_2+6a_3+\ldots$? If, for instance, the postulation is $=a_1+3a_2+6a_3+\ldots-1$, then this would be $=\tfrac{1}{2}n(n+3)-3$, and there would be four asyzygetic curves $X=0$, $Y=0$, $Z=0$, $W=0$. I believe this to be impossible, but the only proof which I can offer rests upon a remark in regard to the form of the tables, pp. 148–149, of my paper "On the Rational Transformation between Two Spaces," *Proc. Lond. Math. Soc.*, t. III. (1870), pp. 127–180. I recall that the Jacobian curve $J(X, Y, Z)=0$ consists of a_1' lines, a_2' conics, a_3' cubics, ... &c., each passing a certain number of times through the $(a_1, a_2, a_3 \ldots)$ points, and that the number of times of passage is shown by these tables; thus, p. 148, $n=5$, $a_1=8$, $a_4=1$: the

Jacobian consists of eight lines and a quartic, and we have the table ($a_1' = 8$, $a_4' = 1$),

	$a_1 = 8$	$a_2 = 0$	$a_3 = 0$	$a_4 = 1$
$a_1' = 8$	1			1
$a_2' = 0$				
$a_3' = 0$				
$a_4' = 1$	8			1^3

showing that the quartic passes through the eight points a_1, and through the point a_4 three times (has a_4 for a triple point). Imagine a new function W. Then in like manner $J(X, Y, W) = 0$ consists of eight lines and a quartic, and this quartic passes through the eight points a_1 and the point a_4 three times; that is, the two quartics intersect in $8+3.3, = 17$ points; and thus the two quartics must be one and the same curve; this implies a syzygy between X, Y, Z, W, viz., W is a mere linear function of X, Y, Z. The general remark is that if in the tables m^p is reckoned as mp^2, then in the table for the several lines (exclusive of those for which the outside accented letter is $= 0$, and therefore the tabular numbers of the line are each $= 0$), *i.e.*, for the lines which correspond to a line, a conic, a cubic, a quartic, &c., respectively, the sums of the tabular numbers are 1^2+1, 2^2+1, 3^2+1, 4^2+1, &c., respectively. This is, in fact, the case for each of the eleven tables of pp. 148–149.

Sir Robert Nicholas Fowler, Bart., M.P., Alderman of the City of London, was of Quaker parentage, being the only son of the late Mr. Thomas Fowler, of Tottenham, a banker in the City. He was born at Tottenham on September 14th, 1828, and was educated at University College, London, where he had a distinguished career. He took his B.A. degree in 1848, when he came out seventh in Classical Honours, and second in Mathematics and Natural Philosophy (the Prizeman being Mr. R. B. Batty, subsequently of Emmanuel College, Cambridge, second Wrangler and second Smith's Prizeman in 1853, and the third man being his cousin, W. Fowler, formerly M.P. for the Borough of Cam[illegible]). He proceed[illegible] M.A. in 1850, and was chosen a Fel[illegible]iversity C[illegible] 1864, and subsequently a member of the Senate of the London University. His death, which was somewhat sudden, took place on May 22nd, 1891. Sir Robert Fowler was elected a Member of the Society March 19th, 1866. See the "Obituary" of the *Times* for May 23rd, 1891.

Dr. John Casey was born in Kilbenny, co. Cork, in May, 1820, and received his early education first at a small school in his native village, afterwards at a larger one near Mitchelstown, and from the beginning showed a decided tendency for mathematics. While still a young man he became a teacher under the Board of National Education, and after a few years was appointed headmaster of the Central Model Schools, Kilkenny, where, having some leisure time at his disposal, he devoted himself to the study of his favourite science. After several years in Kilkenny, a friend drew his attention to Poncelet's celebrated theorem, then exciting much interest in the mathematical world. Knowing little more than the most rudimentary principles of modern geometry, he yet discovered for himself the whole science thereof, and succeeded in giving a purely geometrical proof of the above theorem, the solution of which was the means of introducing him to the notice of Dr. Salmon and Professor Townsend, both of whom began a correspondence with him, and at their suggestion he entered Trinity College in 1858. In College he obtained a sizarship in 1859, a science scholarship in 1861, and his degree in 1862, and in 1869 had the honour of having conferred on him, by the University of Dublin, the degree of LL.D., *honoris causâ*, a distinction also conferred by the Royal University in 1885. In September, 1883, Dr. Casey was selected for the Chair of Higher Mathematics in the Catholic University, and in October of the same year he was offered a Professorship of Mathematics in Trinity College. Though feeling most honoured at the distinction thus offered him, and very grateful to the Fellows and Professors of Trinity for their uniform kindness towards him, he preferred to remain in the Catholic University, the Board of which, on hearing of the sacrifice he had made, immediately raised his salary to the amount offered him by Trinity College, and, in the following December, appointed him also Professor of Mathematical Physics. In 1881, on the foundation of the Royal University, he was appointed to one of the Fellowships, a position he held until his death. In 1866, he was elected a Member of the Royal Irish Academy, and some years later was selected a Member of the Council, acting on the Committee of Science; he was also a Vice-President for a term of five years, and was awarded by the Academy a Cunningham Gold Medal, 1878, in which year he was also chosen as one of the

Secretaries to the Mathematical Section of the British Association, then meeting in Dublin. In 1875 he was elected a Fellow of the Royal Society, in 1878 a Member of the Scientific Society of Brussels, in 1884 a Member of the Société Mathématique de France, and in 1887 a Corresponding Member of the Royal Society of Sciences, Liège.

Professor Casey's contributions to mathematical literature have gained for him a world-wide reputation. In addition to the memoirs some of which are named below, he wrote a series of text-books, which are well known, not only in the United Kingdom, and on the Continent, but also in America and in India. They include the *Sequel to Euclid* (now in its fifth edition), the *Elements of Euclid*, *Elementary Trigonometry*, *A Treatise on Trigonometry*, *A Treatise on Spherical Trigonometry*, and *A Treatise on Analytical Geometry*. Of the *second* edition of this last book (in manuscript) he had passed 400 pages through the press at the time of his death; this work will be completed in a few months, and will extend to about 540 pages.*

Dr. Casey was elected a member of the Society November 12th, 1874. He did not contribute any papers to its *Proceedings*, but acted as an occasional referee for the Council. From 1862 to 1868 he was a co-editor of the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, and contributed a few short Notes to its pages. He also acted for several years as Dublin correspondent for the *Jahrbuch über die Fortschritte der Mathematik.*

Dr. Casey died January 3rd, 1891, in Dublin.

The Royal Society's Catalogue of scientific papers gives some eighteen titles of papers written by Dr. Casey.

The more important papers are—

(1) "Memoir on Bicircular Quartics" (1869).

(2) "Memoir on Cyclides and Sphero-Quartics" (1871).

(3) "Memoir on a new Form of Tangential Equation" (1877).

(4) "Two Memoirs on the Equations of Circles" (1866 and 1878).

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* The above account was drawn up by a friend of Dr. Casey who knew him well, and was communicated [illegible]ociety by Dr. Benjamin Williamson, F.R.S. It agrees in all essential [illegible]he "obitu[illegible]" (signed "G. F. F. G."), in the *Proceedings of t*[illegible], Vol. x[illegible].

(5) "Cubic Transformations" ("Cunningham Memoirs," 1880).

(6) "On the Harmonic Hexagon of a Triangle" (1886).

These and shorter papers were published in the *Transactions and Proceedings of the Royal Irish Academy*, the *Quarterly Journal of Mathematics*, *Hermathena*, *Messenger of Mathematics*, *Annali di Matematica*, *Philosophical Transactions*, or in the *Proceedings of the Royal Society*.

Mr. Henry Martyn Jeffery was born January 5th, 1826, at Lamorran, near Truro, his grandfather, the Rev. W. Curgenven, being rector of the parish. His father, Mr. John Jeffery, of Gwenants, was born at Gwennap, 10th March, 1798, and died at Falmouth, 18th January, 1874; his mother was Laura Curgenven (born at Lamorran, 16th September, 1804, died at Cheltenham, June, 1864). Mr. Jeffery was educated at the Falmouth Classical School (1833–1840), and subsequently under Rev. J. H. Evans, at Sedbergh Grammar School (1841–1845); he entered at St. John's College, Cambridge, on May 12th, 1845, and, after keeping two terms, he migrated, in March, 1846, to St. Catherine's Hall. He graduated B.A. in 1849, as Sixth Wrangler, and Second Class in the Classical Tripos. His private tutor, the late Bishop of Carlisle, writes: "While he (H. M. Jeffery) was reading with me I was nominated Moderator, and was, of course, bound to give up all pupils who would be examined in the Senate-house while I held the office. Jeffery was in this position, and I handed him over to Mr. Griffin (now Canon Griffin). I believe I am right in saying that Mr. Griffin thought highly of his powers." Dr. Goodwin goes on further to say: "Concerning his mathematical powers, I never had any doubt; he was one of the hardest-headed mathematicians with whom I had any dealings in Cambridge."

After taking his degree he took up Hebrew, and was bracketed first Tyrwhitt Scholar in 1852,* in which year also he took his M.A.

Previously to this, he had, in 1850, been lecturer in the College of Civil Engineers at Putney, under the present Dean of Exeter (Dr. Cowie) as Principal. In February, 1852, he left Salisbury House School, Edinburgh, where he had been second master since October, 1851, in order that he might devote the whole of his time to prepare for the Tyrwhitt Scholarship Examination. Dr.

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* "Gained mainly by his skill in composition, to which his previous classical training (entirely abandoned after first year of residence) had adapted him."

Humphreys, the headmaster of the school, was so favourably impressed by his abilities that he desired that Mr. Jeffery should be associated with him as second master in the Cheltenham Grammar School, when he himself was elected to the headmastership in that year (June, 1852). When the headmastership was vacated by Dr. Hayman (June, 1868), Mr. Jeffery was appointed his successor. When the British Association met at Cheltenham, in 1856, Mr. Jeffery was one of the local secretaries: "the public discussions first developed his latent energies, and inclinations for independent inquiry." It was at this meeting he contributed papers: "On a Theorem in Combinations," and on "A Particular Class of Congruences." In 1854, Dr. Humphreys published (1) *Exercitationes Iambicæ;* (2) *Lyra Hellenica, or Translations into Greek;* and in 1855, (3) *Manual of Greek and Latin Prose Composition.* To (1) Mr. Jeffery contributed sixty short introductory exercises (pp. 43–81); to (2) Appendix II. (pp. 186–215); and to (3) Appendix I. (pp. 141–160). Other non-mathematical writings are—"On the best Means of connecting Primary with Intermediate Education" (*Transactions, Social Science Congress at Cheltenham,* 1879, pp. 396–404); "Two sets of Unpublished Letters of the Rev. Henry Martyn, B.D., edited, with prefatory remarks, by his grand-nephew, H. M. J." (1883); "Extracts from the Religious Diary of Miss L. Grenfell" (1890).

Mr. Jeffery was elected a Member of the Society January 14th, 1875, and a Fellow of the Royal Society in 1880. In this year, or in 1881, he left Cheltenham and retired to Falmouth, with the hope strong upon him of collecting his memoirs into two set treatises on "Tangential Coordinates" and "Spherical Analytical Geometry."

> "Many of his pupils have attained high distinction at the Universities and in the various competitive examinations for admission into the public service; and not a few have imbibed their master's passionate love for true and sound knowledge. Every one, in his judgment, should carefully note and follow, at the opening, the bent of his genius."

The following account of the general scope of his mathematical work is taken from the *Biograph,* Vol. VI., p. 515 (second six months of 1881, and so written in Mr. Jeffery's lifetime):—

> "Algebraic curve lines are classified either by the number of points in which they are intersected by a straight line, or by the number of tangents which can be drawn to them from an exterior point; these are severally called order-quantics and class-quantics. Order-cubics have been enumerated by Newton, Plücker, and Cayley; order-quartics partially by Waring, and recently by German mathematicians. To the investigation of class-cubics, both plane and spherical, Jeffery devoted his intervals of leisure for four years (1876–1880), and published the

instalments, as they were completed, in the *Quarterly Journal of Mathematics*. It may be added that he has commenced the study of class-quartics and published the first chapter of the subject (1881)."

The list of his papers in the Royal Society's Catalogue is a long one, and seems to have been entirely restricted to (1) the *Quarterly Journal* and to (2) the *Proceedings* of this Society; in (1) commencing with Vol. IV. (1861), and in (2) with Vol. XII. (1880–81).

In a letter, dated June 17th, 1891, Mr. Jeffery writes:—"In my memoir 'On the Genesis of Binodal Quartic Curves from Conics,' . . . . I expressed (in § 1) a hope to explain the Classification of Binodal Quartics. The accompanying memoir (communicated November 12th, 1891) realizes that hope." This and a paper "On certain Analogous Properties of the Circumscribed and Inscribed Quadrilateral and Pentahedron," in the recently issued October number of the *Quarterly Journal* (No. 100), close the long line of Mr. Jeffery's papers.

Mr. Jeffery died on the 3rd November, 1891.

Cf. also the *Bibliotheca Cornubiensis* of Messrs. Boase and W. P. Courtney (p. 1245, 1874–1882), and the Supplement (p. 422, 1890).

Mr. William Joseph Curran Sharp, of Trinity College, Dublin, was elected a member December 11th, 1879. He was elected a Scholar of Trinity College in 1874, took his B.A. degree in 1875, and proceeded M.A. in 1878. He gained Bishop Law's Prize in 1876.

Mr. Sharp contributed to the *Proceedings* of this Society "The Invariants of a certain Orthogonal Transformation, with special reference to their use in the Theory of the Strains and Stresses of an Elastic Solid" (Vol. XIII.), and three papers on "Simplicissima" (Vols. XVIII., XIX., XXI.); he contributed also to the *Quarterly Journal of Mathematics*.

The following errata have been detected:—

Vol. XXI.

p. 429, after line 5, *for* x, *read* s.

p. 430, on line 15, *for* x, *read* s.

p. 431, on line 34, *for* 8, *read* 10.

p. 432, *for* first four lines, *read*—"then for a uniform train of waves this effect depends on phase; while, for the actual case of changing trains, it would simply diminish the sharpness of the observations by an equal broadening on both sides of the mean of the order ϵ^2, without affecting the mean itself."

p. 454, last line, *for* 1887, *read* 1867.

Vol. XXII.

p. 188, line 15, *for* Q_n^1, *read* ζQ_n^1.

p. 189, line 17, *for* $(\eta^2-\lambda^2c^2)$, *read* $(\eta^2-\lambda^2c^2)^{\frac{1}{2}}$.

p. 189, line 18, *for* $(m+1!)^m$, *read* $m+1!$

p. 190, line 16, *for* μ^4P_n, *read* $\mu^4P_n^1$.

p. 207, line 18, *for* $v = -\frac{h}{\varpi}\cdot\frac{d\psi}{d\beta}$, *read* $v = -\frac{h}{\varpi}\cdot\frac{d\psi}{d\alpha}$.

p. 211, last line, *for* $-\frac{3}{x}-\frac{9}{x^2}$, *read* $1-\frac{3}{x}-\frac{9}{x^2}$.

p. 216, Title *should be* "On Node and Cusp-Loci, which are also Envelopes."

p. 224, lines 11, 12, *dele*.

p. 225, lines 2–4, *dele*, and *read*—

$$\text{“}f_2 = R\,(c_2-\gamma_1)(c_2-\gamma_2)(c_3-\gamma_2)\,;$$

therefore f_2 is of the third order of small quantities, since

$$\gamma_1 = \gamma_2 = \gamma_3 = c_1 = c_2$$

at points on the node-locus. The most important term in $\frac{\partial^2 f_1}{\partial c_1^2}$ is

$$2R\left[(c_1-\gamma_1)+(c_1-\gamma_2)+(c_1-\gamma_3)\right].$$

It is, therefore, of the first order of small quantities at points on the node-locus. Hence, at points on the node-locus, $f_2\Big/\frac{\partial^2 f_1}{\partial c_1^2}$ and $f_2\Big/\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^2$ both vanish, but $f_2\Big/\left(\frac{\partial^2 f_1}{\partial c_1^2}\right)^3$ is finite."

p. 225, lines 10–13, *for* f_1 (alone) in numerator, outside the [ ], *read* f_2.

p. 225, line 14, *for* single f_1, *read* f_2 (three times); after this line (14) *add*—"The last term has been shown to be finite."

Dele p. 226 and lines 1, 2 of p. 227.

p. 227, line 15, *follow with* "(This point is specially considered in Example 4 below.)" and *dele* ( ) below.

p. 227, line 16, *for* single f_1, *read* f_2.

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